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Abstract

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FUNCTORS IN CATEGORIES OF BANACH SPACES

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The present note is devoted to the study of functors in categories of Banach spaces. In particular, the duality of functors and the ring of operators of a functor are investigated. The definition of duality of functors in the category of Banach spaces is a special case of the definition given in the note ([1]); however, here an independent exposition is given. To read the note, familiarity with the concepts of category theory is not necessary.

By a space in what follows we shall mean a Banach space; by a mapping, or operator, a bounded linear operator. The space of operators acting from a space X into a space Y will be denoted by $H(X, Y)$.

By a category of Banach spaces we shall here mean simply a certain collection of Banach spaces.* We assume that all categories under consideration contain the one-dimensional space I. Categories which, together with every space A, contain the conjugate space A' to it will be called regular categories.

We shall say that in a category of Banach spaces $\mathfrak{A}$ a functor** F is given if to each space X of the category $\mathfrak{A}$ there is assigned a Banach space $F(X)$, and to each continuous linear operator φ acting from X to Y $(X, Y \in \mathfrak{A})$ there is assigned a continuous linear operator $F(\varphi)$ acting from $F(X)$ to $F(Y)$ in such a way that the following conditions are satisfied: 1) to the identity mapping $\iota_X : X \to X$ there corresponds the identity mapping $F(\iota_X) : F(X) \to F(X)$; 2) $F(\varphi \cdot \psi) = F(\varphi) \cdot F(\psi)$; 3) $F(\lambda\psi + \mu\psi) = \lambda F(\varphi) + \mu F(\psi)$; 4) $\|F(\varphi)\| \leq \|\varphi\|$ (φ and ψ are operators, λ and μ are numbers).

We say that a mapping α of a functor F into a functor G is given if for every space $X \in \mathfrak{A}$ a mapping $\alpha_X : F(X) \to G(X)$ is defined, and moreover $G(\varphi)\alpha_X = \alpha_{YF}(\varphi)$ for any mapping $\varphi : X \to Y$ from one space of the category $\mathfrak{A}$ into another, and the $\|\alpha_X\|$ are bounded in the aggregate. The norm of the mapping $\alpha : F \to G$ is called

$$\sup_{X \in \mathfrak{A}} \|\alpha_X\|.$$

By defining in the natural way addition of mappings and multiplication by a number, we turn the set of mappings of the functor F into the functor G into a Banach space $H(F, G)$.

In what follows, unless the contrary is stated, we assume that the functors under consideration are defined in an arbitrary category $\mathfrak{A}$.

Example 1. Assigning to each space $X \in \mathfrak{A}$ the Banach space $H(A, X)$ (A is a fixed Banach space), and to a mapping $\varphi : X \to Y$ the mapping $\tilde{\varphi} : H(A, X) \to H(A, Y)$, defined by the formula $\tilde{\varphi}(a) = \varphi \cdot a$, we obtain a functor; we shall denote this functor by Ω_A.

Example 2. Assigning to each space $X \in \mathfrak{A}$ the tensor product $A \widehat{\otimes} X$ ([3]), and to a mapping $\varphi : X \to Y$ the mapping $\iota_A \widehat{\otimes} \varphi$, where $\iota_A : A \to A$ is the identity mapping, we obtain the functor Σ_A.

* Using the terminology of category theory, one may say that we consider categories whose objects are Banach spaces, while the set of morphisms from an object X to an object Y is the set of all bounded linear mappings from X to Y.

** The usual definition of a functor ([2]), acting from the category $\mathfrak{A}$ to the category of all Banach spaces, does not contain conditions (3) and (4).

Example 3. Let a normed space of numerical sequences n satisfy the following conditions: a) the sequences $e_i(0, \dots, 0, 1, 0, \dots) \in n$ and $\|e_i\|_n = 1$; b) if $x(x_1, \dots, x_i, \dots) \in n$ and $y(y_1, \dots, y_i, \dots)$ is such a sequence that $|y_i| = |x_i|$, then $y \in n$ and $\|y\|_n \leqslant \|x\|_n$. Then with its help one can define a certain functor n, namely, to each space $X \in \mathfrak{A}$ there is assigned the space $n(X)$ of such sequences $(x_1, \dots, x_i, \dots)$ of elements of the space X that the sequence of norms $(\|x_1\|, \dots, \|x_i\|, \dots) \in n$; the norm in the space $n(X)$ is defined by the formula

$$\|(x_1, \dots, x_i, \dots)\| = \|(\|x_1\|, \dots, \|x_i\|, \dots)\|_n;$$

the mapping $n(\varphi)$ is constructed coordinatewise.

A mapping α of a space A into a space B gives rise to a mapping Σ_α of the functor Σ_A into the functor Σ_B ($(\Sigma_\alpha)_X = \alpha \widehat{\otimes} \iota_X$). One can prove that mappings of the form Σ_α exhaust the mappings of the functor Σ_A into the functor Σ_B; in other words,

$$H(\Sigma_A, \Sigma_B) = H(A, B).$$

If $A \in \mathfrak{A}$, then the following relation (1) holds:

$$H(\Omega_A, F) = F(A).$$

Definition. We shall say that a functor G is dual to a functor F (we write $G = DF$) if for every space $A \in \mathfrak{A}$ we have

$$G(A) = H(F, \Sigma_A)$$

and for every mapping $\varphi : A \to B$ ($A, B \in \mathfrak{A}$) the mapping

$$G(\varphi) : G(A) = H(F, \Sigma_A) \to G(B) = H(F, \Sigma_B)$$

is induced by the mapping $\Sigma_\varphi : \Sigma_A \to \Sigma_B$.

Example 1. $D\Sigma_A = \Omega_A$.

Example 2. If the category $\mathfrak{A}$ contains the space A, then

$$D\Omega_A = \Sigma_A.$$

Example 3. In a regular category

$$Dl^p = l^q \quad (p^{-1} + q^{-1} = 1),$$

where l^p is the functor corresponding to the space l^p of sequences summable with the p-th power.

For every functor F there is a naturally constructed mapping

$$\varkappa : F \to DDF \quad (1).$$

The functor F is called reflexive if the mapping $\varkappa$ is an isometry of the functors F and DDF.

Theorem 1 ([1]). *For any functors F and G the spaces $H(F, DG)$ and $H(G, DF)$ are isometric. If G is a reflexive functor, then the spaces $H(F, G)$ and $H(DG, DF)$ are isometric.*

Let $\mathfrak{A}$ be a category of Banach spaces containing the space A and its conjugate space A'; let F be a functor in the category $\mathfrak{A}$. Define a mapping

$$\lambda_A : DF(A) \to (F(A'))'$$

as follows: if

$$\alpha \in DF(A) = H(F, \Sigma_A),$$

then $\lambda_A(\alpha)$ is the functional on $F(A')$ which assigns to each element $x \in F(A')$ the number $v\alpha_{A'}$ (here

$$v : \Sigma_A(A') = A \widehat{\otimes} A' \to I$$

is the scalar product:

$$v(a \otimes a') = (a, a')$$

). The mapping λ_A is an isometry ([1]). In what follows we shall identify the space $DF(A)$ with the isometric subspace $\lambda_A DF(A)$ of the space $(F(A'))'$.

The mapping

$$F : X = H(I, X) \to H(F(I), F(X))$$

in view of the relation

$$H(X, H(F(I), F(X))) = H(X \widehat{\otimes} F(I), F(X))$$

gives rise to a mapping

$$\gamma_X : X \widehat{\otimes} F(I) \to F(X).$$

We shall call a functor F, acting in the category $\mathfrak{A}$, a functor of type Σ if for every space $X \in \mathfrak{A}$ the image of the mapping γ_X is everywhere dense in $F(X)$.

Let $\mathfrak{R}$ be a regular category consisting of reflexive spaces.

Theorem 2. *In the category $\mathfrak{R}$, for every functor F and every finite-dimensional space A the relation*

$$DF(A) = (F(A'))'$$

holds (more precisely, the mapping λ_A is an isometry "onto"). If F is a functor of type Σ, then the relation

$$DF(A) = (F(A'))'$$

is valid for every space A.

Let F be a functor in the category $\mathfrak{R}$, containing the space X and the space $c_0(X)$ (by c_0 is denoted the space of numerical sequences converging to zero with metric

$$\|(x_1, \dots, x_i, \dots)\| = \sup |x_i|;$$

by λ_j denote the mappings $c_0(X) \to X$, defined by the formulas

$$\lambda_j(x_1, \dots, x_i, \dots) = x_j$$

). The sequence $t_1, \dots, t_j, \dots$ of elements

a space $F(X)$ will be called special if there exists an element $t \in F(c_0(X))$ such that $t_j = F(\lambda_j)(t)$. Define on $F(X)$ the special topology as the strongest of the topologies in which all special sequences converge to zero.

Theorem 3. If $a \in DF(A)$, then the mapping $\alpha_X : F(X) \to A\hat{\otimes}X$ takes every special sequence into a weakly convergent sequence.

Theorem 4. If F is a functor in the category of all separable spaces, then the space $DF(I)$ is isometric to the subspace of the space $(F(I))'$ consisting of all functionals on $F(I)$ that are continuous in the special topology.

Define, for every functor F in the category $\mathfrak{A}$, the mapping $\mu_A : DF(A) \to \Omega_{A'}DF(I)$ by putting $\mu_A(\alpha) = \beta$, where β assigns to the element $\varphi \in A'$ the composition of functor mappings $\alpha : F \to \Sigma_A$ and $\Sigma_\varphi : \Sigma_A \to \Sigma_I$.

We shall say that the space A satisfies condition $(*)$ if there exists a directed family $\{P_\lambda\}$ of finite-dimensional operators, $P_\lambda : A \to A$, such that $\sup \|P_\lambda\| < \infty$ and $\lim_\lambda(\varphi, P_\lambda x) = (\varphi, x)$ for all $x \in A$, $\varphi \in A'$.

Theorem 5. For every functor F in the category $\mathfrak{A}$ and every space $A \in \mathfrak{A}$ satisfying condition $(*)$, the mapping $\mu_A : DF(A) \to \Omega_{A'}DF(I)$ is a monomorphism (i.e. sends only the zero element to zero). Under the same conditions the mapping $\nu_A : DF(A) \to \Omega_{F(I)}(A)$, defined by the formula $\nu_A(\alpha) = \alpha_I$, is a monomorphism*.

With the help of the results formulated above one can find dual functors for various concrete functors. The following proposition makes it possible to determine the dual functor for the functor n generated by the sequence space n.

Theorem 6. In a regular category containing the space c_0, we have $Dn(X) = \bar{n}(X)$, where $\bar{n}$ is the space of numerical sequences with norm

$$\|(\lambda_1, \dots, \lambda_i, \dots)\|_{\bar{n}} = \sup(\lambda_1 \mu_1 + \dots + \lambda_i \mu_i + \dots)$$

(the least upper bound is taken over all possible sequences $(\mu_1, \dots, \mu_i, \dots) \in n$ having norm $\|(\mu_1, \dots, \mu_i, \dots)\|_n = 1$); the space $\bar{n}$ consists of all sequences for which

$$\|(\lambda_1, \dots, \lambda_i, \dots)\|_{\bar{n}} < \infty.$$

Every reflexive functor in the category of Banach spaces is the projective limit of a spectrum of functors of the form Σ_A (see ([1])). Therefore, apparently, it is reasonable to consider reflexive functors in the category of Banach spaces as generalized Banach spaces; in doing so the functor Σ_A should be regarded as the functor corresponding to the space A.

The basic concept in the theory of Banach spaces—the concept of a linear bounded operator acting from one space into another, as we saw above, also has meaning for functors. To each functor F there corresponds a ring of operators acting in it (the Banach space $H(F, F)$ is a normed ring with respect to composition of operators). The ring of operators of the functor Σ_A is isomorphic to the ring of operators of the space A. An antihomomorphism D is naturally defined from the ring of operators of the functor F into the ring of operators of the functor DF; if F is a reflexive functor, then D is an anti-isomorphism of these rings of operators.

Theorem 7. Let the functor n in the category $\mathfrak{R}$, containing the spaces c_0 and l^1, be defined by means of the sequence space n. Then every operator T acting in the functor n is uniquely determined by some infinite numerical matrix (t_{ij}) (if $x' = T_X(x)$,

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* In the formulation of this theorem, condition $(*)$ may be replaced by Grothendieck' s "approximation condition" ([3]), p. 165.

$x, x' \in X,\ x = (x_1, \dots, x_i, \dots),\ x' = (x_1, \dots, x_i', \dots)$, then

$$x_i' = \sum_{j=1}^{\infty} t_{ij} x_j.$$

The matrix (t_{ij}) defines an operator in the functor $\mathfrak{n}$ if and only if the matrix $(|t_{ij}|)$ generates a bounded operator in the space $\mathfrak{n}$.

In other words, an operator acting in the space $\mathfrak{n}$ extends to an operator acting in the functor $\mathfrak{n}$ if and only if it is representable as the difference of two operators that are monotone with respect to the cone of nonnegative sequences.

It turns out to be possible to introduce the notion of a Hilbert functor, analogous to the notion of a Hilbert space.

Let us note that the mapping

$$k_{XY} : F(X) \hat{\otimes} \hat{\otimes} DF(Y) \to X \otimes Y$$

is naturally defined and plays the role of a generalized scalar product (if $\alpha \in F(X)$, $\beta \in DF(Y)$, then the mapping $\beta_X : F(X) \to X \hat{\otimes} Y$ is defined and one may put

$$k_{X,Y}(\alpha \otimes \beta) = \beta_X(\alpha)).$$

The functor F is called Hilbertian if: 1) the functors F and DF are isometric (an isometry $\lambda : F \to DF$ is fixed); 2) for any $a \in F(X)$, $b \in F(Y)$ we have

$$k_{X,Y}(a \otimes \lambda_Y b) = k_{Y,X}(b \otimes \lambda_X a);$$

3) the mapping

$$h_X : F(X) \hat{\otimes} F(X) \to X \hat{\otimes} X,$$

defined by the formula

$$h_X = k_{X,X}(\iota \otimes \lambda_X),$$

is monotone (in the space $X \hat{\otimes} X$ a partial ordering is introduced: the cone of nonnegative elements is taken to be the closure of the set of elements of the form

$$\sum_{i=1}^{n} x_i \otimes x_i,$$

ι is the identity mapping; $X \hat{\otimes} Y$ is naturally identified with $Y \hat{\otimes} X$).

The ring of operators of a Hilbert functor F turns out to be a normed ring with involution (the involution is defined by the formula

$$T^* = \lambda^{-1}(DT)\lambda).$$

Thanks to this one can speak of self-adjoint unitary projection, etc. operators in the functor F. One can also define the notion of a positive operator: an operator T is called positive if

$$h_X(Tx \otimes x) \geq 0$$

for every $x \in F(X)$, i.e. if the mapping

$$h_X(T \hat{\otimes} \iota)$$

is monotone. For every operator U the product U^*U is a positive self-adjoint operator.

An example of a Hilbert functor is the functor l^2 (if this functor is considered in the regular category).

Remark. The definitions and theorems formulated above pertain to Banach spaces over the field of real numbers. For Banach spaces over the field of complex numbers it is convenient to change the definition of duality, considering instead of the functor $\widetilde{\Sigma}_A$ the functor Σ_A, given by the formula

$$\Sigma_A(X) = \hat{A}\hat{\otimes}X,$$

where A is a semilinear space isomorphic to the space A ([4], Ch. II, Appendix, no. 2). All the main results are then preserved (sometimes in a somewhat modified formulation).

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Note: Figure translations are in progress. See original paper for figures.