M. M. Gekhtman

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Abstract

Full Text

M. M. Gekhtman

On the Principle of Limiting Amplitude

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Mathematics

Consider the Cauchy problem

$$\frac{\partial^2 v(x,t)}{\partial t^2} + Lv = -f(x)e^{i\omega t}; \quad (1)$$

$$v(x,0) = \frac{\partial v(x,0)}{\partial t} = 0. \quad (2)$$

Here $\omega > 0$; $f(x)$ is a finite sufficiently smooth function; L is a homogeneous positive definite elliptic operator with constant coefficients of order $2m$; $x = (x_1, \dots, x_n)$ is a point of the n-dimensional space R_n.

The Cauchy problem (1)–(2) is posed correctly; therefore there exists its unique solution $v(x,t)$.

If there exists, uniformly in x varying in each bounded domain of R_n, the limit $u(x)$ equal to

$$u(x) = \lim_{t\to\infty} v(x,t)e^{-i\omega t}, \quad (3)$$

then this limit is called the **limiting amplitude** ([1]).

The purpose of the present note is to study the behavior of the solution $v(x,t)$ of the Cauchy problem (1)–(2) when t is sufficiently large and x varies in a bounded domain. In particular, the question of the existence of the limiting amplitude is considered.

In order to simplify the calculations, we shall assume that n is an odd number and $L \equiv (-1)^m \Delta^m$, where Δ is the Laplace operator. The problem is solved by the method of analytic continuation of the resolvent kernel ([2]).

We seek the solution of problem (1)–(2) by means of the Laplace transform with respect to t, putting

$$w(x, \lambda) = \int_0^\infty v(x, t) e^{-\lambda t}\, dt \qquad (\operatorname{Re}\lambda > 0).$$

Then for $w(x, \lambda)$ we obtain the equation

$$Lw = -\lambda^2 w + \frac{f(x)}{\lambda - i\omega}. \tag{4}$$

In $L^2(R_n)$ equation (4) has a unique solution w, and for $\operatorname{Re}\lambda > 0$

$$w(x, \lambda) = \int_{R_n} H(x, y, \lambda) \frac{f(y)}{\lambda - i\omega}\, dy; \tag{5}$$

$H(x, y, \lambda)$ is the resolvent kernel of the operator L. We shall show that $H(x, y, \lambda)$, as a function of λ, admits analytic continuation into the domain $-a \le \operatorname{Re}\lambda \le 0$ $(a > 0)$, except, possibly, for the point $\lambda = 0$, and we assume that a cut has been made along the negative real semi-axis.

We shall seek $H(x, \lambda)$ as a decreasing solution of the equation

$$(L + \lambda^2) P(D) H_0(x, \lambda) = \delta(x); \tag{6}$$

here $P(D) = [(-1)^p \Delta^p + 1]$; $\delta(x)$ is the Dirac function; $H(x, \lambda) = P(D) H_0(x, \lambda^2)$, and the number p is chosen from the condition $n - 1 < 2m + 2p$.

Expanding $\delta(x)$ into plane waves ([3]) and using the Fourier transform, we obtain

$$H(x, \lambda) = P(D) \int_\Omega \left[\int_{-\infty}^{\infty} \frac{s^{n-1} e^{isz}\, ds}{(s^{2m} + \lambda^2)(s^{2p} + 1)} \right] d\omega. \tag{7}$$

In formula (7), Ω is the unit sphere, $\omega = (\omega_1, \dots, \omega_n)$ is a point on this sphere, and

$$z = \sum_{i=1}^{n} x_i \omega_i.$$

By virtue of formula (5),

$$w(x, \lambda) = \int_{R_n} H_0(x, y, \lambda) \frac{P(D) f(y)}{\lambda - i\omega}\, dy. \tag{8}$$

Here we have used the fact that $f(y)$ is a sufficiently smooth finite function, and $H_0(x, y, \lambda)$ depends on the difference $x - y$.

Applying the residue theorem, one can explicitly write the expression

$$H_0(x,\lambda) = \int_\Omega \left[\frac{\pi i}{m} \sum_{j=0}^{m-1} \frac{-\varepsilon_j^n \lambda^{(n-2m)/m} e^{i|z|\varepsilon_j \lambda^{1/m}}}{\lambda^{2p/m}\varepsilon_j^{2p}+1} + \frac{\pi i}{p} \sum_{k=0}^{p-1} \frac{-\delta_k^n e^{i|z|\delta_k}}{\lambda^2+\delta_k^{2m}} \right] d\omega. \quad (9)$$

In this formula $\varepsilon_j = \sqrt[2m]{-1}$, $\operatorname{Im}\varepsilon_j > 0$, $j = 0, 1, \ldots, m-1$; analogously, $\delta_k = \sqrt[2p]{-1}$ are defined.

From formulas (8) and (9) it follows that $w(x,\lambda)$ admits an analytic continuation into the region $\operatorname{Re}\lambda \le 0$, except, perhaps, for the point $\lambda = 0$, and at the point $\lambda = i\omega$, $w(x,\lambda)$ has a simple pole with residue equal to 1. Along the negative real half-axis in the λ-plane a cut is made.

If $a > 0$ is fixed, then p can be chosen so large that

$$|w(x,\lambda)| \le \frac{c(x)}{|\lambda|^2} \quad \text{for large } |\lambda|. \quad (10)$$

Theorem 1. Let n be an odd number, and let L be a homogeneous elliptic operator with constant coefficients of order $2m$. In order that the limiting amplitude exist, it is sufficient that the kernel of the resolvent of the operator L, $H(x,\lambda)$, in a neighborhood of zero (with respect to the variable λ) satisfy the estimate

$$|H(x,\lambda)| \le \frac{c(x)}{|\lambda|^{1-\varepsilon}} \quad (\varepsilon > 0). \quad (11)$$

The proof is carried out by the method of contour integration in the same way as in ([2]), except that the contour of integration must not contain the point $\lambda = 0$ inside it.

From this theorem we obtain consequences.

Let $n > 2m$. Then the kernel of the resolvent admits an estimate of the form

$$|H(x,\lambda)| \le c(x), \quad \text{if } |\lambda| \text{ is small.}$$

Consequently, the limiting amplitude exists. This is the case considered in ([4]).

Let $m < n < 2m$. In this case, when computing the resolvent kernel one may set $p = 0$. We obtain

$$H(x,\lambda) = c_0 \lambda^{(n-2m)/m} \sum_{j=0}^{m-1} -\varepsilon_j^n \int_0^1 (1-p^2)^{(n-3)/2} e^{i\varepsilon_j \lambda^{1/m}|x|p}\, dp,$$

$$|H(x,\lambda)| \leq \frac{c(x)}{|\lambda|^{2-n/m}};$$

since $n > m$, condition (11) is satisfied; the limiting amplitude exists, and we have obtained a strengthening of the results of [5].

Let $n = m$. In this case

$$v(x,t) = e^{i\omega t} \int_{\mathring{R}_n} H(x,y,i\omega) f(y)\,dy + c_0 \int_{\mathring{R}_n} f(y)\,dy + o(1);$$

$o(1)$ is a quantity tending to zero as $t \to \infty$, $c_0 \neq 0$. In order to preserve the limiting-amplitude principle, it is necessary additionally to require that

$$\int_{\mathring{R}_n} f(y)\,dy = 0.$$

If $m > n$, then one may also take $p = 0$. $H(x,\lambda)$, as a function of λ, has at zero a singularity of type $1/\lambda^{\gamma}$, where $1 < \gamma < 2$. Expanding $e^{i\varepsilon_j \lambda^{1/m}|x|}$ in a series, we obtain a finite number of terms whose singularity in λ at zero will be of the form $1/|\lambda|^{1+\varepsilon}$ ($\varepsilon \geq 0$), while the remaining terms lead to the already considered singularity of the form $1/|\lambda|^{1-\varepsilon_1}$ ($\varepsilon_1 > 0$). Only the terms of the first type will give an asymptotic behavior growing as $t \to \infty$.

Let us consider, for example, the asymptotic behavior corresponding to the first term of the expansion. Here the growth in t will be maximal. Clearly, this term has the form $H_1(x,\lambda) = c/\lambda^{\gamma}$.

Carrying out the calculations, we obtain:

$$v(x,t) = e^{i\omega t} \int_{\mathring{R}_n} H(x,y,i\omega) f(y)\,dy + c_0 t^{1+\gamma} \int_{\mathring{R}_n} f(y)\,dy + \cdots + o(1).$$

Here terms are omitted which, as functions of t, grow more slowly than $t^{1+\gamma}$. Thus it has been proved:

Theorem 2. *Let n be an odd number. In order that a limiting amplitude exist for a homogeneous positive definite elliptic operator L, it is necessary and sufficient that $n > m$.*

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.