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Abstract

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EXTENSION OF AN EQUIMORPHISM TO INFINITY

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Will every equimorphism of a space induce a topological mapping on the sphere of infinitely distant points? It is easy to see that this assertion is not true for Euclidean space E^n, $n \geq 2$ (even* if, instead of equimorphisms, one considers only L-morphisms—mappings satisfying the two-sided Lipschitz condition $l \cdot xy \leq {}'x'y \leq L \cdot xy$; ${}'x, {}'y$ are the images of arbitrary points x, y; l, L are constants). It is noteworthy that in Lobachevsky-Bolyai space such an induction to infinity does take place; see the induction theorem of § 3. This theorem may also be formulated otherwise: an equimorphism $f : H^n \to {}'H^n$ of hyperbolic spaces is represented in the Beltrami–Klein model by an equimorphism (but, of course, not conversely). It should be noted that an equimorphism of the model representing an L-morphism f need not itself be an L-morphism (for example, ${}'\xi = \xi$, ${}'\eta = \eta + 1$, where ξ, η are Cartesian coordinates, $ds^2 = \operatorname{ch}^2 \eta \, d\xi^2 + d\eta^2$, on the plane H^2).

§ 1. **The image of a line under an equimorphism.** Under an equimorphism $f : H^n \to {}'H^n$ of hyperbolic spaces, the following assertions hold:

Theorem 1. *The image ${}'\Gamma$ of any rectilinear ray Γ has an asymptotic direction.*

Theorem 2. *The image ${}'\Gamma$ deviates throughout its entire extent by a bounded amount from every line G of the direction asymptotic to ${}'\Gamma$.*

The proofs of these theorems become quite transparent if one restricts oneself only to L-morphisms. Meanwhile every equimorphism of geodesic spaces essentially (i.e., for not too small distances) satisfies a Lipschitz condition ([1]). Here (§ 2) these theorems are proved in detail only for L-morphisms. In proving them for arbitrary equimorphisms, one essentially follows the same path, but with the aid of more careful estimates, with integrals replaced by sums (or series), and the lengths of, generally speaking, unrectifiable arcs by the perimeters of inscribed broken lines with segments of bounded length.

§ 2. **Proof of Theorems 1 and 2 for L-morphisms.** As is known ([2]), under L-morphisms rectifiable arcs go over into rectifiable arcs, and for their lengths the Lipschitz condition with the same constants is satisfied. In differential form, $l\,ds \leq d's \leq L\,ds$. It is therefore not difficult to see that in this case each line goes over into a line for which the ratio of the length of any arc to the contracting chord is bounded: $\widetilde{xy}/xy \leq \lambda = L/l$.

Proof of Theorem 1 for $n = 2$. Let o be the origin of the ray Γ; r, σ the polar coordinates of the point $'x = f(x)$, with the pole taken to be the point $'o = f(o)$. Let us compute the total variation of the angle σ as $'x$ moves along $'\Gamma$ from some fixed point $'x_0 \neq {}'o$ to infinity. Denote by s the length of the segment of the curve $'\Gamma$ from $'o$ to $'x$ $(s = {}'o'x,\ s_0 = {}'o'x_0)$. Since

$$ds^2 = dr^2 + \operatorname{sh}^2 r\, d\sigma^2, \tag{1}$$

we have $|d\sigma| \leq ds/\operatorname{sh} r$.

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* For E^2 it suffices to consider a mapping which, in polar coordinates r, φ, for $r \geq 1$ is given by the formulas $'r = r,\ {}'\varphi = \varphi + \log r$.

Noting that $s/r \ll \lambda$, we obtain:

$$\text{total variation of the angle} = \int_{s=s_0}^{\infty} |d\sigma| \leq \int_{s_0}^{\infty} \frac{ds}{\operatorname{sh} r} \leq \int_{s_0}^{\infty} \frac{ds}{\operatorname{sh} s/\lambda} = -\lambda \log \operatorname{th} \frac{s_0}{2\lambda}. \tag{2}$$

Thus the total variation is bounded; consequently, for sufficiently large s_0 it is arbitrarily small, whence it follows that σ tends to a limiting value as x recedes along Γ' to infinity. This limiting value of σ also determines the asymptotic direction γ' of the "ray" Γ'.

The proof for arbitrary n follows from the same formulas (1), (2), in which $d\sigma$ must now be understood as the central projection from o' of the arc element ds onto the unit sphere described about o'. We note that the proof remains valid if an arbitrary fixed point is taken for the pole o'.

Remark. Theorem 1 remains valid for any Riemannian space of spherical symmetry,

$$ds^2 = dr^2 + f^2(r) d\sigma^2,$$

provided that

$$\int^{\infty} \frac{dr}{f(r)}$$

converges absolutely.

Proof of Theorem 2. We take the line G as the axis oz of a "cylindrical" coordinate system in H^n. For any point $x \in H^n$, denote by r its distance from oz, and by z the projection of the vector $\overrightarrow{ox}$ onto oz. Then

$$ds^2 \geq \operatorname{ch}^2 r\, dz^2 + dr^2.$$

Along Γ' the coordinates z and r will be continuous functions $z(s), r(s)$ of the arc length s, measured from some fixed point $x_0 \in \Gamma'$. We shall prove that $r(s)$ is bounded. If this is not so, the following possibilities must be considered: $1^\circ, 2^\circ, 3^\circ$.

1°. $r(s)$ tends monotonically to $+\infty$. Then

$$\frac{r(s) + |\Delta z| + r(s + \Delta s)}{\Delta s} > \frac{1}{\lambda},$$

but

$$\Delta s \geq |\Delta z| \operatorname{ch} r(s),$$

therefore

$$|\Delta z| < \lambda \frac{2r(s) + \Delta r}{\operatorname{ch} r(s) - \lambda},$$

as soon as $\operatorname{ch} r(s) > \lambda$. Let s_1 be one of such values of s: $\operatorname{ch} r(s_1) > \lambda$. Consider a sequence $s_1 < s_2 < \cdots$ such that $r(s_{k+1}) = r(s_k) + 1$, and denote $r(s_k) = r_k$, $z(s_k) = z_k$; then

$$|\Delta z_k| < \lambda \frac{2r_k + 1}{\operatorname{ch} r_k - \lambda},$$

$$|\Delta z| \leq \sum_1^\infty |\Delta z_k| < \lambda \sum_1^\infty \frac{2r_k + 1}{\operatorname{ch} r_k - \lambda} < +\infty,$$

which is impossible, since the axis oz has an asymptotic direction.

2°. The case in which $r(s)$ tends to $+\infty$ nonmonotonically reduces to the preceding one by replacing $r(s)$ by

$$r^*(s) = \min_{s \leq t < +\infty} r(t).$$

3°. $r(s)$ does not tend to $+\infty$; then, consequently, there exists a sequence $s_k \to +\infty$ such that $r(s_k) < c = \text{const}$; a simple calculation shows that c may be taken equal to* $c_2 = \operatorname{ar ch} 2\lambda$, and the length $|\Delta z|$ of each** interval of variation of $z(s)$, where $r(s) \geq c_2$, does not exceed $2c_2$; therefore

$$r(s) \leq (2\lambda + 1)c_2 = (2\lambda + 1) \operatorname{ar ch} 2\lambda$$

for all sufficiently large s. This proves Theorem 2.

Remark. If the line G of the asymptotic direction is drawn through the initial point o' of the ray Γ', then the last inequality is valid **for all** s, and in this case the deviation $r(s)$ is bounded not only along Γ', but also over the totality of all Γ'.

* In fact it suffices to put $c = c_1 + \alpha$, where $\operatorname{ch} c_1 = \lambda$, and α is an arbitrarily small positive constant.

** Except, perhaps, the first.

Corollary. Each line Δ under the mapping f passes into a line $'\Delta = f(\Delta)$, having two distinct infinitely distant points $'u \neq {}'v$. Along its entire length $'\Delta$ deviates from the line $'u'v$ by a bounded amount (the latter bound is bounded also in the aggregate of all $'\Delta$).

§ 3. The induction theorem. An equimorphism $f : H^n \to {}'H^n$ of hyperbolic spaces induces a topological mapping $\varphi : S^{n-1} \to {}'S^{n-1}$ of their infinitely distant spheres; f, φ together define a homeomorphism $\tilde{f} : \widetilde{H}^n \to {}'\widetilde{H}^n$, where $\widetilde{H}^n, {}'\widetilde{H}^n$ denote the extensions of the spaces $H^n, {}'H^n$ obtained in the usual way by adjoining infinitely distant points.

Proof. From a fixed point $o \in H^n$ draw rays Γ to each point $u \in S^{n-1}$. Denote by $'u$ the infinitely distant point of the image $'\Gamma = f(\Gamma)$. This defines the function φ: $\varphi(u) = {}'u$.

1) The one-to-one character of φ follows from Theorem 2: if two distinct points u_1 and $u_2 \in S^{n-1}$ were mapped to the same point $'u \in {}'S^{n-1}$, then the corresponding rays Γ_1 and Γ_2 would have to pass into lines $'\Gamma_1, {}'\Gamma_2$, deviating in a bounded way from the ray $'o'u$; thus they themselves would deviate from one another in a bounded way, which is impossible.

2) The continuity of φ (as also of $\tilde{f}$) follows from the remark to Theorem 2. Let us prove at once the continuity of $\tilde{f}$ at a point $u \in S^{n-1}$; for this purpose consider a sequence $x_k \in \widetilde{H}^n$ tending to u. If the sequence $'x_k = \tilde{f}(x_k)$ did not tend to $'u = \varphi(u)$, then from it one could extract a subsequence tending to another point $'v \neq {}'u$, $'v \in {}'S^{n-1}$. Changing notation, we shall suppose that $'x_k$ is already this indicated subsequence. Thus, $'x_k \to {}'v \neq {}'u$, $x_k \to u$.

Denote by Γ_k (or Γ) the ray issuing from o and passing through x_k (through u); suppose, moreover, that u_k is the infinitely distant point of the ray Γ_k. On each Γ_k one can choose a point y_k so that $|oy_k| \to \infty$, $\rho(y_k, \Gamma) \to 0$; then $\rho('y_k, {}'\Gamma)$

must also tend to zero. However, this is impossible, since $'\Gamma_k$ deviates in a bounded way (see the remark to Theorem 2) from $'ou_k$, while $'\Gamma$ deviates in a bounded way from $'o'u$ (Theorem 2), and the angle $'v'o'u_k \to 0$. The continuity of $\tilde{f}$ is proved.

3) From 1) and 2) and from the compactness of $\widetilde{H}^n$ it follows that $\tilde{f}$ is a homeomorphism.

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CITED LITERATURE

1. V. A. Efremovich, *UMN*, **8**, 5 (1953).

2. V. A. Efremovich, *Uch. zap. Ivanovsk. ped. inst.*, **5**, 3 (1954).

Note: Figure translations are in progress. See original paper for figures.