ON CONFORMAL MAPPINGS OF DOMAINS WITH CORNERS AND ON THE DIFFERENTIAL PROPERTIES OF SOLUTIONS OF THE POISSON EQUATION IN DOMAINS WITH CORNERS

1963

SovietRxiv

Abstract

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MATHEMATICS

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ON CONFORMAL MAPPINGS OF DOMAINS WITH CORNERS AND ON THE DIFFERENTIAL PROPERTIES OF SOLUTIONS OF THE POISSON EQUATION IN DOMAINS WITH CORNERS

(Presented by Academician S. L. Sobolev on 17 IV 1963)

We introduce the following notation. Let Γ be a rectifiable Jordan curve situated in the complex plane $z = x + iy$, and let its equation be written in the form $z(s) = x(s)+iy(s)$, where s is the arclength of Γ, measured from some fixed point P_0 on Γ. The curve $\Gamma \in C^{\bar r+\alpha}$ ($\bar r \geq 1$ an integer, $0 < \alpha < 1$) if $z(s) \in C^{\bar r+\alpha}[0, l]$, i.e., the function $z(s)$ has $\bar r$ derivatives and the $\bar r$-th derivative satisfies a Hölder condition with exponent α on the segment $0 \leq s \leq l$, where l is the length of Γ. If Γ is a closed Jordan curve and $z(s)$, considered as a periodic function of period l, belongs to the class $C^{\bar r+\alpha}[0, 2l]$, then we shall agree to say that $\Gamma \in C_*^{\bar r+\alpha}$, and that the domain g bounded by Γ belongs to the class $C^{\bar r+\alpha}$. If at the point P_0 the curve forms an angle ω, $0 < \omega < 2\pi$, and $\Gamma \in C^{\bar r+\alpha}[0, l]$, then, by definition, $g \in C_\omega^{\bar r+\alpha}$; if, moreover, both pieces of Γ adjacent to the corner point P_0 are straight-line segments, then we shall say that $g \in C_\omega^{\bar{\bar r}+\alpha}$; if these pieces have contact of order r with their tangents at P_0, then, by definition, $g \in C_\omega^{0^{\bar r}+\alpha}$. We shall assume that the corner point P_0 is situated at the origin and that one of the pieces of Γ is tangent at the point $(0, 0)$ to the positive semiaxis x.

The following theorem of Kellogg ([1,2]) on conformal mappings of smooth domains is known.

Theorem 1 (Kellogg). *If a domain $G_1 \in C^{\bar r+\alpha}$ and a domain $G_2 \in C^{\bar r+\alpha}$, then any function $w = f(z)$ realizing a conformal mapping of the domain G_1 onto the domain G_2 satisfies the conditions*

$$|f'(z)| > c > 0 \quad \text{for } z \in \bar G_1, \qquad f(z) \in C^{\bar r+\alpha}(\bar G_1)^*.$$

From Kellogg' s theorem it is not difficult to derive a corollary. Let the function $w = f(z)$ realize a conformal mapping of a bounded domain G_1 with boundary Γ, containing an arc $\gamma_1 \in C^{\bar r+\alpha}$, onto a bounded domain G_2 with boundary containing an arc $\gamma_2 \in C^{\bar r+\alpha}$, and suppose that γ_1 is carried into γ_2; then for any $\varepsilon > 0$ and any domain G_1^ε such that $G_1^\varepsilon \subset G_1$ and $\rho(G_1^\varepsilon, \Gamma \setminus \gamma_1) > \varepsilon$, the conditions

$$f(z) \in C^{\bar{r}+\alpha}(\bar{G}_1^{\varepsilon}), \qquad |f'(z)| > C_{\varepsilon} > 0 \quad \text{for } z \in G_1^{\varepsilon}.$$

hold.

For functions realizing a conformal mapping of a domain with an angle ω onto a domain with the same angle, we have proved the following theorem, analogous to Kellogg' s theorem.

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* That is, in the domain G_1, $f^{(\bar{r})}(z)$ is bounded and satisfies a Hölder condition with exponent α.

Theorem 2. Let one of the following conditions be satisfied:

a) the domain $G_1 \in C_{\omega}^{\bar{r}+\alpha}$, the domain $G_2 \in C_{\omega}^{\bar{r}+\alpha}$, $0 < \omega < 2\pi$, $\omega \neq \pi$, $\bar{r}+\alpha < 1+\pi/\omega$;

b) $G_1 \in C_{\omega}^{\bar{r}+\alpha}$, $G_2 \in C_{\omega}^{\bar{r}+\alpha}$, $\omega = \pi/j$ $(j = 2, 3, ...)$, $\bar{r}$ arbitrary.

Then every function $w = f(z)$ realizing a conformal mapping of the domain G_1 onto the domain G_2 and carrying a corner point of the domain G_1 to a corner point of the domain G_2, satisfies the conditions

$$f(z) \in C^{\bar{r}+\alpha}(G_1), \qquad |f'(z)| > c > 0 \quad \text{for } z \in G_1.$$

The condition $|f'(z)| > c > 0$ is contained in one of the theorems of Lichtenstein ([3],[4]) concerning conformal mappings of domains with corners. With the aid of Theorem 2, the theorems formulated in ([5]) concerning the behavior of harmonic functions prescribed in domains $G \in C_{\omega}^{\bar{r}+\alpha}$ are generalized.

Theorem 3. Let $0 < \omega < 2\pi$, $r < \pi/\omega + 1/p$, the domain $G \in C_{\omega}^{\bar{k}+\alpha}$ $(\bar{k}+\alpha > r + 1/p)$. Then, if on the boundary of the domain G a function $f(s) \in H_p^r(M, [0, l])$, $1 < p \leq \infty$, $p > \omega/\pi$, is prescribed, and $f(0) = f(l)$ in the case $r > 1/p$, there exists a harmonic function u in the domain G, satisfying the conditions

$$u\big|_{\Gamma} = f, \qquad u \in H_p^{r+1/p}(cM, G)\,*;$$

such a function is unique; the converse theorem is true.

Theorem 4. Let $\omega = \pi/j$ $(j = 2, 3, ...)$, the domain $G \in C_{\omega}^{0\,\bar{k}+\alpha}$, $\bar{k}+\alpha > r+1/p$, $1 < p \leq \infty$. Then, if on the boundary of G a function $f(s) \in H_p^r(M, [0, l])$ is prescribed, satisfying the conditions

$$\left.\frac{d^{kj} f(s)}{ds^{kj}}\right|_{s=0} = (-1)^k \left.\frac{d^{kj} f(l-s)}{ds^{kj}}\right|_{s=0} \quad \text{for all integers } k < \frac{r-1/p}{j},$$

then there exists a function u harmonic in G such that

$$u\big|_\Gamma = f, \qquad u \in H_p^{r+1/p}(cM, G);$$

the converse theorem is also true.

With the aid of Theorems 3 and 4, theorems are proved on the differential properties of solutions of the Poisson equation with zero boundary conditions.

Theorem 3′**.** Let $0 < \omega < 2\pi$, $r + 2 < \pi/\omega + 2/p$, the domain $G \in C_\omega^{\bar{k}+\alpha}$, $\bar{k} + \alpha > r + 2$. Then, if $F(x, y) \in H_p^r(M, G)$, $1 < p \leq \infty$, there exists in the domain G a function u satisfying the conditions

$$\Delta u = F, \qquad u\big|_\Gamma = 0, \qquad u \in H_p^{r+2}(cM, G); \tag{1}$$

the converse theorem is true.

Theorem 4′**.** Let $\omega = \pi/j$ $(j = 2, 3, ...)$, the domain $G \in \overset{0}{C}{}_\omega^{\bar{k}+\alpha}$, $\bar{k} + \alpha > r + 2$. Then, if $F(x, y) \in H_p^r(M, G)$, $1 < p \leq \infty$, and at the corner point the compatibility conditions are fulfilled

$$\sum_{k=2}^{mj} \sum_{i=0}^{\left[\frac{k-2}{2}\right]} (-1)^i C_{mj}^k \cos^{mj-k} \omega \, \sin^k \omega \, \frac{\partial^{mj-2} F(x, y)}{\partial x^{mj-k+2i} \, \partial y^{k-2-2i}} \Bigg|_{\substack{x=0 \\ y=0}} = 0$$

for all integers m such that $0 \leq mj - 2 < r - 2/p$, then there exists u satisfying (1); the converse theorem is also true.

* Definition of the classes $H_p^r(M, [0, l])$ and $H_p^r(G)$ see ([8]).

One can construct examples showing that Theorems 2-4, 3′, 4′ cannot be strengthened in the terms of these theorems. We note the works ([6,7]), which are related to the questions touched upon.

Let us also note that, as examples show, all the conditions entering into Theorems 2, 4, 4′ are essential; failure of even one of them (for example, if $\omega \neq \pi/j$; or if the compatibility conditions on the derivatives f or F are not satisfied; or if $g \in \widetilde{C}_\omega^{k+\alpha}$, and not in $\overset{0}{C}{}_\omega^{k+\alpha}$) leads to the falsity of the theorem.

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Received
9 IV 1963

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Note: Figure translations are in progress. See original paper for figures.