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Abstract

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ON THE CONNECTION BETWEEN SMALL CHANGES OF A SYSTEM OF DIFFERENTIAL EQUATIONS AND THE CORRESPONDING POINT MAPPING

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In the present paper the following problem is solved.

Let T be a point mapping of the hyperplane S into itself, generated by the phase trajectories of the system of differential equations

$$\frac{dx_i}{dt} X_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n), \tag{1}$$

defined in some domain G, in which the right-hand sides are sufficiently smooth functions*. Suppose that the point mapping T has a fixed point M^*. The phase trajectory Γ passing through the point M^* is closed; it corresponds to a periodic solution of the system of equations (1)

$$x_i = \varphi_i(t) \quad (i = 1, 2, \ldots, n) \tag{2}$$

of some period τ.

The question is whether every small change of the mapping T in a neighborhood of M^* can be obtained by equally small changes of the right-hand sides of the differential equations.

Here, by small changes we mean changes that are small together with their partial derivatives up to some order m ($m \geqslant 1$).

The solution of this problem is of interest in the study of the dependence of periodic motions on parameters ($^{1-4}$). As is known, small changes of the right-hand sides of equations (1) correspond to equally small changes of the point mapping T. In view of this, the question of bifurcations of a periodic solution under small additions to the right-hand sides of the differential equations, if these additions do not destroy the existence of the point mapping T, reduces to the study of bifurcations of fixed points of the transformation under its small changes (^{4,5}). However, with such an approach the following question remains open: whether every possible case of dependence of the fixed point of the mapping on small changes of it corresponds to the corresponding case of dependence of the periodic motion on small changes of the right-hand sides of the differential equations. This question is answered by the following theorem.

Theorem. *Let T be a point mapping of the hyperplane S into itself, generated by the phase trajectories of the differential equations (1); let M^* be its fixed point, and let the closed phase trajectory Γ, passing through the point M^*, intersect the hyperplane S without being tangent to it. Then, in a small neighborhood of the point M^*, for sufficiently small δ, the point mapping*

$$\overline{T} = T + \delta\Omega_0, \tag{3}$$

where Ω_0 is any mapping continuously differentiable m times, can be generated by phase trajectories of differential equations of the form

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \dots, x_n) + \omega_i(x_1, \dots, x_n; \delta), \tag{4}$$

where $\omega_i(x_1, \dots, x_n; \delta)$ are functions continuously differentiable m times with respect to $x_1, x_2, \dots, x_n$, vanishing for $\delta = 0$ together with their first m partial derivatives.

* Here and below, by sufficiently smooth functions it is enough to mean functions continuously differentiable $2m$ times.

We pass to new variables $u_1, u_2, \dots, u_n$, putting

$$x_j = \varphi_j(u_1) + \alpha_{j2}(u_1)u_2 + \dots + \alpha_{jn}(u_1)u_n, \tag{5}$$

where φ_j are the same functions as in equations (2), and $\alpha_{js}(u_1)$ are sufficiently smooth functions of u_1, periodic with period τ, chosen so that, for $u_1 = 0$, the point $x_1, \dots, x_n$ lies in the hyperplane S. In addition, of course, it is assumed that the Jacobian of the change of variables (5) is nonzero in some neighborhood of the closed phase curve Γ.

Denote by T_u the point mapping of the hyperplane $u_1 = 0$ into the hyperplane $u_1 = u$, generated by the phase trajectories of equation (1). This mapping is the identity for $u = 0$, and for $u = \tau$ coincides with the mapping T; moreover, its Jacobian does not vanish in some sufficiently small neighborhood of the phase curve Γ.

Consider the mapping

$$\widetilde{T}_{u_1} = T_{u_1} + \delta\Phi(u_2^2 + \dots + u_n^2)\Psi_0(u_1)\Omega_0 + \Psi_1(u_1)\Omega_1 + \dots + \Psi_m(u_1)\Omega_m, \tag{6}$$

where $\Phi(r)$ is a function continuously differentiable m times, equal to one for $r \leq r_0 > 0$ and equal to zero for $r \geq r_1 > r_0$; $\Psi_0(u_1), \dots, \Psi_m(u_1)$ are functions continuously differentiable $m+1$ times and satisfying the conditions

$$\Psi_j^{(s)}(+0) = 0, \qquad \Psi_j^{(s)}(\tau - 0) = \delta_j^s \quad (s, j = 0, 1, 2, \dots, m); \tag{7}$$

$$\Omega_s = \left(\frac{\partial^s T_{u_1}}{\partial u_1^s}\right)_{u_1=+0} \left(T_{u_1} + \delta\Phi(u_2^2 + \dots + u_n^2)\Omega_0\right)^{-1}_{u_1=\tau-0} - \left(\frac{\partial^s T_{u_1}}{\partial u_1^s}\right)_{u_1=\tau-0} \quad (s = 1, 2, \dots, m). \tag{8}$$

This point mapping $\widetilde{T}_{u_1}$ is continuously differentiable m times and coincides with T_{u_1} for $u_2^2 + \dots + u_n^2 \geq r_1$, and for $\delta = 0$ passes into the mapping T_{u_1}. For sufficiently small δ, the Jacobian of the mapping $\widetilde{T}_{u_1}$ is nonzero.

Construct a system of differential equations (u is the vector with components $u_2, u_3, \dots, u_n$)

$$\frac{du}{du_1} = \frac{\partial \widetilde{T}_{u_1}}{\partial u_1} \widetilde{T}_{u_1}^{-1} u = F(u, u_1). \tag{9}$$

These equations are defined for $0 \leq u_1 \leq \tau$ and are continuously differentiable m times with respect to $u_1, u_2, \dots, u_n$. We extend them in u_1, proceeding from the requirement that the right-hand sides be periodic in u_1 with period τ. For $u_2^2 + \dots + u_n^2 \geq r_1$, the extended equations (9) will coincide with the original equations (1), written in the variables $u_1, u_2, \dots, u_n$.

For $\delta = 0$ these equations pass into equations (1). For the complete proof of Theorem 1 it remains to establish that the extension of equation (9) is continuously differentiable m times with respect to $u_1, \dots, u_n$. By the construction of the transformation $\widetilde{T}_{u_1}$, identically with respect to $u_2, u_3, \dots, u_n$ we have the relations

$$\left(\frac{d^s u}{du_1^s}\right)_{u_1=+0} = \left(\frac{d^s u}{du_1^s}\right)_{u_1=\tau-0} \quad (s = 1, 2, \dots, m). \tag{10}$$

Indeed,

$$\frac{d^s u}{du_1^s} = \frac{\partial^s \widetilde{T}_{u_1}}{\partial u_1^s} \widetilde{T}_{u_1}^{-1} u;$$

further, according to (6) and (7),

$$\left(\frac{\partial^s \widetilde{T}_{u_1}}{\partial u_1^s}\right)_{u_1=+0} = \left(\frac{\partial^s T_{u_1}}{\partial u_1^s}\right)_{u_1=+0}, \qquad \left(\frac{\partial^s \widetilde{T}_{u_1}}{\partial u_1^s}\right)_{u_1=\tau-0} = \left(\frac{\partial^s T_{u_1}}{\partial u_1^s}\right)_{u_1=\tau-0} + \delta\Phi\Omega_s,$$

and therefore (10) holds by virtue of (8).

From relations (10) for $s = 1$ there follows the continuity of the right-hand sides of the continued equations (9) with respect to the variables $u_1, u_2, \dots, u_n$ and continuous differentiability up to order m inclusive with respect to the variables $u_2, u_3, \dots, u_n$.

From relation (10) for $s = 2$, in view of the fact that

$$\frac{d^2u}{du_1^2} = \frac{\partial F}{\partial u_1} + \frac{\partial F}{\partial u} F(u, u_1),$$

it follows that

$$\left(\frac{\partial F}{\partial u_1}\right)_{u_1=+0} = \left(\frac{\partial F}{\partial u_1}\right)_{u_1=\tau-0}$$

identically with respect to $u_2, u_3, \dots, u_n$. Therefore F is continuously differentiable with respect to u_1, and $\partial F/\partial u_1$ is $m-1$ times continuously differentiable with respect to $u_2, u_3, \dots, u_n$. Similarly, step by step, relations (10) establish the m-fold continuous differentiability of the right-hand side of equation (9) with respect to the variables $u_1, u_2, \dots, u_n$.

The point mapping T in a neighborhood of the fixed point M^* can be written in the form

$$\bar{u}_j = f_j(u_2, u_3, \dots, u_n) \qquad (j = 2, 3, \dots, n). \tag{11}$$

If all the roots of its characteristic equation at the point M^* are different from unity, then the periodic motion Γ is called simple. In this case any small change of the differential equations (1) leads to an equally small change of the periodic solution (2). On the contrary, in the presence of roots equal to unity, small changes of the right-hand sides of equations (1) may cause the disappearance of the periodic motion or its splitting into several periodic motions.

Thus, in the simplest case of one unit root, after a corresponding change of variables the transformation T can be written in the form

$$\bar{v}_1 = v_1 + \Psi_1(v_1, \dots, v_{n-1}),$$

$$\bar{v}_j = z_{jv}j + \Psi_j(v_1, \dots, v_{n-1}) \qquad (j = 2, 3, \dots, n-1), \tag{12}$$

where $\Psi_1, \Psi_2, \dots, \Psi_{n-1}$ are of at least second order of smallness with respect to $v_1, v_2, \dots, v_{n-1}$, and $z_2, z_3, \dots, z_n$ are different from unity. To determine the fixed points of this transformation (12), we have the system of equations

$$\Psi_1(v_1, \dots, v_{n-1}) = 0, \qquad (z_j - 1)v_j + \Psi_j(v_1, \dots, v_{n-1}) = 0. \tag{13}$$

After substituting the expressions for $v_2, \dots, v_{n-1}$ through v_1, found from the last $n - 2$ equations (13), into the first equation, we arrive at an equation with one unknown v_1. If $v_1 = 0$ is its s-fold root, then for any $\bar{s}$ lying between 0 and s $(0 \leq \bar{s} \leq s)$ for even s, and between 1 and s $(1 \leq \bar{s} \leq s)$ for odd s, there exist such small changes of the transformation T, and consequently, by Theorem 1 $(m \geq s + 1)$, also such small additions to the right-hand sides of equations (1), under which $\bar{s}$ periodic motions* with period close to τ appear in a neighborhood of Γ.

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* In the case of the phase plane $(n = 2)$ this result was established in (2); for arbitrary n and $\bar{s} = 2$, in (3); for $n = 3$ and $\bar{s} = s$ it was formulated in (6).

Note: Figure translations are in progress. See original paper for figures.