Yu. M. Berezanskii, S. G. Krein, Ya. A. Roitberg

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

Yu. M. Berezanskii, S. G. Krein, Ya. A. Roitberg

A THEOREM ON HOMEOMORPHISMS AND LOCAL INCREASE OF SMOOTHNESS UP TO THE BOUNDARY OF SOLUTIONS OF ELLIPTIC EQUATIONS

(Presented by Academician I. G. Petrovskii on 21 VIII 1962)

In this note a theorem is proved on homeomorphisms realized by an elliptic operator, for the case of general homogeneous boundary conditions, and its application is given to the local increase of smoothness up to the boundary of generalized solutions that are elements of spaces with negative norm.

Theorems on homeomorphisms for spaces of functions of increased smoothness follow from coercivity inequalities, which for general boundary conditions were obtained by M. Shekhter ([2–4]). For the case of the first boundary-value problem, a collection of theorems on homeomorphisms, also pertaining to spaces of functions of decreased smoothness, was obtained by Lions (see ([1,5])) and developed in ([6,7]). Below, for the general case, results of the same type are obtained, and thereby a complete collection of homeomorphisms is established. The proof is also based on the interpolation theorem and on general considerations expressed in a report by one of the authors at the Fourth Conference on Functional Analysis (Baku, 1959).

In considering the question of smoothness of generalized solutions, the complete collection of homeomorphisms is used.

1. Let H_0 be a Hilbert space and let D be an unbounded self-adjoint positive operator with domain $\mathcal{D}(D)$ such that $\|u\|_0 \leqslant \|Du\|_0$. The domain of definition $\mathcal{D}(D^\alpha)$ $(\alpha > 0)$ is a Hilbert space H_α with respect to the scalar product

$$(u, v)_\alpha = (D^\alpha u, D^\alpha v)_0.$$

We take this space as positive; the corresponding negative space $H_{-\alpha}$ may be obtained by completing H_0 with respect to the scalar product

$$(f, g)_{-\alpha} = (D^{-\alpha} f, D^{-\alpha} g)$$

$(f, g \in H_0)$; H_α and $H_{-\alpha}$ are mutually conjugate with respect to the scalar product $(\cdot, \cdot)_0$. The family $\{H_\alpha\}$ $(-\infty < \alpha < \infty)$ is called a Hilbert scale of spaces.

The following interpolation theorem holds:

Let $\{H_\alpha\}$ and $\{H'_\alpha\}$ be two Hilbert scales and let B be a linear operator, continuously acting from H_{α_0} to H'_{α_1} and from H_{β_0} to H'_{β_1}. Then B acts continuously from any $H_{\alpha_0(\mu)}$ to $H'_{\alpha_1(\mu)}$,

$$\alpha_i(\mu) = (1-\mu)\alpha_i + \mu\beta_i$$

$(i = 0, 1)$, $0 \leqslant \mu \leqslant 1$. This theorem was obtained independently by Lions ([8]) and by one of the authors ([9]). We shall now give one application of this theorem.

Consider H_0 and a fixed space H_l $(l > 0)$. Let N and N^* be finite-dimensional subspaces of H_l, and let

$$H'_0 = H_0 \ominus N^*, \qquad H''_0 = H_0 \ominus N$$

be their orthogonal complements in H_0; let B be a continuous operator, with respect to the metric of H_0, acting from all of H'_0 onto a dense part of H''_0; let B^* be the operator adjoint to B, acting from H''_0 to H'_0. We shall additionally assume that the ranges of the operators B and B^* are contained in H_l. It is convenient to denote the closures of these ranges in the metric of H_s $(0 \leqslant s \leqslant l)$, respectively, by $H_s(\mathrm{gr})$ and $H_s(\mathrm{gr})^+$. Let $H_{-s}(\mathrm{gr})$ and $H_{-s}(\mathrm{gr})^+$ be the corresponding conjugate spaces with respect to $(\cdot,\cdot)_0$.

Theorem 1. If the inequalities

$$\|Bf\|_l \leq C_1\|f\|_0 \quad (f \in H'_0), \qquad \|B^*g\|_l \leq C_2\|g\|_0 \quad (g \in H''_1), \tag{1}$$

hold, then for $\alpha \in [0, l]$ the estimate

$$\|Bf\|_\alpha \leq C_3\|f\|_{H_{\alpha-l}(\mathrm{rp})^+} \quad (f \in H'_0) \tag{2}$$

is valid.

Let us outline the proof, first restricting ourselves to the case in which there is no defect: $N = N^* = 0$, $H'_0 = H''_0 = H_0$. The second inequality in (1) permits one to regard B^* as a continuous operator from H_0 into H_l; denote by $\widetilde{B}$ the adjoint operator to it, acting from H_{-l} into H_0. The operator $\widetilde{B}$ is an extension of B, and $\|\widetilde{B}f\|_0 \leq C_2\|f\|_{-l}$. Hence, from the first inequality in (1), using the interpolation theorem we obtain the estimate

$$\|Bf\|_\alpha \leq C_1^{\alpha/l}C_2^{1-\alpha/l}\|f\|_{\alpha-l} \quad (f \in H_0).$$

But, as is known, the space adjoint to the subspace $H_s(\mathrm{rp})^+$ of the space H_s is isometric to the quotient space H_{-s} by its subspace V_{-s}, consisting of elements annihilating $H_s(\mathrm{rp})^+$;

$$\|\varphi\|_{H_{-s}(\mathrm{rp})^+} = \inf_{\psi\in V_{-s}} \|\varphi+\psi\|_{-s}.$$

And since the operator $\widetilde{B}$ annihilates elements of V_{-s}, it follows that

$$\|Bf\|_\alpha = \|\widetilde{B}(f+\psi)\|_\alpha \le C_1^{\alpha/l} C_2^{1-\alpha/l} \|f+\psi\|_{\alpha-l} \quad (\psi \in V_{-s}).$$

Passing here to the inf, we find (2). In the case where a defect is present, the operator B^* may be regarded as acting from H_0'' into H_l. The adjoint operator $\widetilde{B}$ will act from H_{-l} into $H_0'' \subset H_0$, and all the preceding arguments can be repeated; the interpolation theorem should be applied to the operator $\widetilde{B}\widetilde{P}$, where $\widetilde{P}$ is the extension to H_{-l} of the operator P of orthogonal projection of H_0 onto H_0'.

2. We describe the general scheme for applying Theorem 1 to differential operators. Let $H_0 = L_2(G)$, where G is a bounded domain in n-dimensional space, and let π_l ($l > 0$ an integer) be the Sobolev space $W_2^l(G)$. It can be shown that the Hilbert scale H_α ($0 \le \alpha \le l$) consists of the Sobolev spaces $W_2^\alpha(G)$ for integral α and of the Aronszajn–Slobodetskii spaces for fractional α; moreover the norms in H_α are equivalent to the corresponding norms in $W_2^\alpha(G)$.

Consider in $H_0 = L_2(G)$ an operator A, generated by an elliptic differential expression $\mathcal{L}$ with sufficiently smooth coefficients and by a certain system of boundary conditions. Let $W_2^l(\text{rp})$ be the totality of all functions in $W_2^l(G)$ satisfying these conditions; $Au = \mathcal{L}u,\ u \in D(A) = W_2^l(\text{rp})$. We shall assume that A^* is constructed analogously from the formally adjoint expression $\mathcal{L}^+$ and the conditions $(\text{rp})^+$. Suppose the following assertions hold: 1) the subspaces N and N^* of solutions of the equations $Au = 0$ and $A^*v = 0$, respectively, are finite-dimensional. Denote

$$H_l(\text{rp}) = W_2^l(\text{rp}) \cap H_0'', \qquad H_l(\text{rp})^+ = W_2^l(\text{rp})^+ \cap H_0',$$

where, as before,

$$H_0' = H_0 \ominus N, \qquad H_0'' = H_0 \ominus N^*;$$

2) the inequalities

$$\|u\|_l \le C_1 \|Au\|_0 \quad (u \in H_l(\text{rp})), \qquad \|v\|_l \le C_2 \|A^*v\|_0 \quad (v \in H_l(\text{rp})^+)$$

hold; 3) the ranges of A on $H_l(\text{rp})$ and of A^* on $H_l(\text{rp})^+$ coincide respectively with H_0' and H_0''.

If we denote by B the operator inverse to the restriction of A to $H_l(\text{rp})$, then the hypotheses of Theorem 1 will be satisfied for it, and from that theorem, for $\alpha \in [0, l]$, one obtains the inequality

$$\|Au\|_{H_{-\alpha}(\text{rp})^+} \ge C\|u\|_{l-\alpha} \quad (u \in H_l(\text{rp})).$$

In many cases, when $\alpha = k \in [0, l]$ is an integer, the converse inequality is also valid; then

$$C\|u\|_{l-k} \le \|Au\|_{H_{-k}(\text{rp})^+} \le C_1\|u\|_{l-k} \quad (u \in H_l(\text{rp}),\ k = 0, \dots, l). \tag{3}$$

This inequality makes it possible to extend, by continuity, the operator A to an operator A_{-k} that carries out a homeomorphic mapping of $H_{l-k}(\mathrm{gr})$ onto all of $H_{-k}(\mathrm{gr})^+$ $(k = 0, \dots, l)$. Precisely such a situation occurs for the operator considered in the next section.

3. We shall now formulate the exact assertion that can be obtained on the basis of the considerations of Section 2. In doing so we shall make essential use of the results of the papers [2–4] of M. Schechter. Thus, in a bounded domain G with boundary Γ, consider a properly elliptic [2] differential expression of order $l = 2m$ with complex coefficients

$$\mathcal{L}u = \sum_{|\mu| \le 2m} a_\mu(x) D^\mu u$$

$$\left(\mu = (\mu_1, \dots, \mu_n),\ |\mu| = \mu_1 + \dots + \mu_n, \quad D^\mu = \left(\frac{1}{i} \frac{\partial}{\partial x_1} \right)^{\mu_1} \cdots \left(\frac{1}{i} \frac{\partial}{\partial x_n} \right)^{\mu_n} \right).$$

On Γ there are given m differential expressions

$$B_k = \sum_{|\mu| \le m_k} b_{k\mu}(x) D^\mu$$

$$(k = 1, \dots, m;\ m_k \le 2m - 1),$$

which determine the boundary conditions. Now $W_2^l(\mathrm{gr})$ consists of the functions $u \in W_2^l(G)$ for which $B_k u = 0$ $(k = 1, \dots, m)$.

Theorem 2. *Let the operators B_k be normal and cover $\mathcal{L}$ in the sense of [2,3]. Consider the operator A: $Au = \mathcal{L}u$ $(u \in H_{2m}(\mathrm{gr}))$. 1) If the boundary Γ is of class C^{4m+s}, the coefficients $a_\mu(x) \in C^{2m+\max(|\mu|,s)}(G \cup \Gamma)$, $b_{k\mu}(x), b_{k\mu}^+(x) \in C^{2m+s-1}(\Gamma)$, then, for an integer $s \ge 0$, the restriction of the operator A to*

$$\dot{H}_{2m+s}(\mathrm{gr}) = H_{2m}(\mathrm{gr}) \cap W_2^{2m+s}(G)$$

establishes a homeomorphism $H_{2m+s}(\mathrm{gr}) \to H_s^+ = W_2^s \cap H_0'$. 2) If the preceding smoothness conditions are satisfied for $s = 0$, then the operator A admits an extension A_{-s} on $H_{2m-s}(\mathrm{gr})$, carrying out a homeomorphism $H_{2m-s}(\mathrm{gr}) \to H_{-s}(\mathrm{gr})^+$ $(0 \le s \le 2m,\ s$ integer). 3) Let the smoothness conditions be the same as in 1); then the operator A admits an extension to the space $\dot{H}_{-s}$, conjugate to $H_s^+ = W_2^s(G) \cap H_0'$; this extension carries out a homeomorphism $\dot{H}_{-s} \to H_{-2m-s}(\mathrm{gr})^+$.

Let us explain that the homeomorphism in the case of the first pair of spaces follows directly from [3,4], and in the case of the second pair from the construction of Section 2; moreover the second of the inequalities (3) is established by means of integration by parts. For the third pair it is derived from the first homeomorphism by passing to conjugate spaces.

Above, the principal difficulty consisted in establishing the homeomorphism for the second pair. For zero boundary conditions this fact is, in essence, known [5–7]. We note that in the latter case, and even in the more general case when the boundary operators B_k have order $\leq m-1$, the homeomorphism can also be established with the aid of the Aronszajn lemma. In this case the energy inequalities are proved in a stronger form—with boundary norms.

4. Theorem 2 on homeomorphisms can be applied to the local increase of smoothness up to the boundary of generalized solutions of elliptic equations that are elements of spaces with negative norm. For simplicity we shall speak about zero boundary conditions and strongly elliptic expressions. Below all indices in the notation of spaces are integers.

Let us give a definition. Let G be a bounded domain, on whose boundary Γ lies a piece Γ_0; let $\varphi \in W_2^s(G)$, $s<0$. We shall say that φ belongs to W_2^t, with $t>s$, inside G up to the piece Γ_0 (we shall write this as $\varphi \in W_{2,\text{loc}}^t(G,\Gamma_0)$), if for every subdomain $G' \subset G$ having common boundary with G only inside the piece Γ_0, there exists $\psi_{G'} \in W_2^t(G)$ such that

$$(\varphi, u)_0 = (\psi_{G'}, u)_0$$

for all $u \in C^\infty(G \cup \Gamma)$ that additionally vanish in a neighborhood of $G \setminus G'$. It holds that

Theorem 3. Let an expression $\mathcal{L}$ of order $l=2m$ be strongly elliptic, let its coefficients $a_\mu \in C^{|\mu|+p}(G \cup \Gamma_0)$, and let Γ_0 be of class C^{l+p} $(p \geqslant 0)$. Suppose that a function $\varphi \in W_2^s(G)$ $(s = \dots, -1, 0, 1, \dots)$ satisfies the equation $\mathcal{L}u = f$ inside G up to the piece Γ_0, where zero boundary conditions are prescribed $\left(D^\alpha u\big|_{\Gamma_0} = 0,\ |\alpha| \leqslant m-1\right)$. Here $f \in W_{2,\text{loc}}^{s-l}(G,\Gamma_0)$; satisfaction of the equation means that

$$(\varphi, \mathcal{L}^+ v)_0 = (f, v)_0 \tag{4}$$

for all $v \in W_2^{2m+\max(-s,0)}(G)$, $D^\alpha v\big|_{\Gamma_0} = 0$ $(|\alpha| \leqslant m-1)$, and additionally vanishing in neighborhoods of the set $\Gamma \setminus \Gamma_0$. Suppose that $f \in W_{2,\text{loc}}^t(G,\Gamma_0)$, where $t > s-l$ is of arbitrary sign. Then: 1) for $p<m$ and $s \in [-p,p)$ we automatically conclude that $\varphi \in W_{2,\text{loc}}^{\min(t+l,p)}(G,\Gamma_0)$; 2) for $p \geqslant m$ and $s \in [-p, p+l)$, that $\varphi \in W_{2,\text{loc}}^{\min(t+l,p+l)}(G,\Gamma_0)$. Moreover, if the minimum turns out to be positive, then φ (more precisely, the function with which φ coincides)

together with a certain number of derivatives vanishes on Γ_0. Namely, in case 1) $D^\alpha \varphi\big|_{\Gamma_0} = 0$ ($|\alpha| \leqslant \min(t+l, p, m) - 1$); in case 2) $D^\alpha \varphi\big|_{\Gamma_0} = 0$ ($\alpha \leqslant \min(t+l, m) - 1$).

In the same way, a simpler result on increasing smoothness inside the domain can be formulated.

The difference between Theorem 3 and a number of earlier results in this direction ([10–12,2]) is that here smoothness is increased up to the boundary for solutions that are generalized functions, and not functions from $L_2(G)$. This circumstance plays an essential role, for example, in various questions of spectral theory. Theorem 3 makes it possible to establish smoothness up to the boundary of generalized eigenfunctions, of the kernel of the resolvent, and the like, for elliptic operators, if it is known in advance that these functions are elements of spaces with negative norm. We note that definition (4) of a generalized solution agrees with the definition arising in the theory of generalized eigenfunctions (see ([13])). We also emphasize that, owing to the local character of Theorem 3, it is applicable to problems in unbounded domains.

Theorem 3 can be established for properly elliptic expressions and other boundary conditions for which Theorem 2 is valid and which satisfy certain broad additional restrictions. In conclusion, we note that the results presented in §§ 3–4, with the aid of the considerations of ([14]), are generalized to equations with discontinuous coefficients.

Institute of Mathematics, Academy of Sciences of the Ukrainian SSR
Voronezh State University
Stanislav Pedagogical Institute

Received
17 VII 1962

REFERENCES

1. L. Gårding, UMN, **15**, no. 1, 137 (1960).

2. M. Schechter, Comm. Pure and Appl. Math., **12**, no. 3, 457 (1959); Sbornik: Mathematics, **4**, no. 5, 1960, p. 92.

3. M. Schechter, Comm. Pure and Appl. Math., **12**, no. 4, 561 (1959); Sbornik: Mathematics, **4**, no. 6, 1960, p. 3.

4. M. Schechter, Ann. of Math., **72**, no. 3, 481 (1960).

5. E. Magenes, C. Stampacchia, Ann. Scuola Norm. Super. Pisa, ser. III, **12**, 247 (1958).

6. J. L. Lions, E. Magenes, Ann. Inst. Fourier, **11**, 137 (1961).

7. J. L. Lions, E. Magenes, Ann. Scuola Norm. Super. Pisa, Sci. fis. e mat., **15**, no. 12, 41 (1961).

8. J. L. Lions, Bull. Math. Soc. Sci. Math. Phys., RPR, **2** (50), no. 4, 419 (1958).

9. S. G. Krein, DAN, **130**, no. 3, 491 (1960).

10. K. O. Friedrichs, Comm. Pure and Appl. Math., **6**, 299 (1953).

11. L. Nirenberg, Comm. Pure and Appl. Math., **8**, 648 (1955).

12. F. E. Browder, Comm. Pure and Appl. Math., **9**, 351 (1956).

13. Yu. M. Berezanskii, Ukrainian Mathematical Journal, **11**, no. 1, 16 (1959).

14. Ya. A. Roitberg, Z. G. Sheftel, DAN, **148**, no. 4 (1963).

Note: Figure translations are in progress. See original paper for figures.