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1963

SovietRxiv

Abstract

Full Text

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ON THE ANALYTIC DESCRIPTION OF CERTAIN GEOMETRIC OBJECTS

(Presented by Academician A. A. Dorodnitsyn, March 21, 1963)

Operations are introduced that make it possible to write the equation of a given geometric figure (drawing). The need for these operations arose for the author in considering certain problems of structural mechanics. B. L. Yushchenko used these operations to construct a new method for solving problems of mathematical (in particular, linear) programming.

§ 1. R-operations and their properties

Let us divide the set of real numbers into two classes: 1) positive numbers and 0, 2) negative numbers. The membership of a number a in these classes will be determined as follows: $S(a) \sim a \geqslant 0$.

Basic R-operations.

Definition 1. The R-negation of a number a $(a \neq 0)$ is the number $b = (-1)a$. The usual notation is $b = -a$. The R-negation of 0 is called 0, i.e. $-0 = 0$.

Definition 2. A number c is called the R-conjunction of the numbers a and b, if

$$c = \frac{a + b - \sqrt{a^2 + b^2}}{2};$$

notation: $c = a \wedge_1 b$.

Definition 3. A number c is called the R-disjunction of the numbers a and b, if

$$c = \frac{a + b + \sqrt{a^2 + b^2}}{2};$$

notation: $c = a \vee_1 b$.

Properties of R-operations.

1°. $-(-a) = a$.
2°. $a \wedge_1 b = b \wedge_1 a$.
3°. $a \vee_1 b = b \vee_1 a$.
4°. $a \wedge_1 b = 0 \sim [S(a) \wedge b = 0] \vee [S(b) \wedge a = 0]$.
5°. $a \vee_1 b = 0 \sim (a = 0 \wedge b = 0) \vee [\overline{S(a)} \wedge b = 0] \vee [\overline{S(b)} \wedge a = 0]$.
6°. $-(a \wedge_1 b) = -a \vee_1 -b$.
7°. $-(a \vee_1 b) = -a \wedge_1 -b$.
8°. $S(a \wedge_1 b) \sim S(a) \wedge S(b)$.
9°. $S(a \vee_1 b) \sim S(a) \vee S(b)$.
10°. $S[-(a \wedge_1 b)] \sim \overline{S(a)} \vee \overline{S(b)}$.
11°. $S[-(a \vee_1 b)] \sim \overline{S(a)} \wedge \overline{S(b)}$.
12°. $S[(a \wedge_1 b) \wedge_1 c] \sim S[a \wedge_1 (b \wedge_1 c)] \sim S[b \wedge_1 (a \wedge_1 c)]$.
13°. $S[(a \vee_1 b) \vee_1 c] \sim S[a \vee_1 (b \vee_1 c)] \sim S[b \vee_1 (a \vee_1 c)]$.
14°. $S[a \vee_1 (b \vee_1 c)] \sim S(a \vee_1 b) \vee S(a \wedge_1 b)$.
15°. $S[a \vee_1 (b \wedge_1 c)] \sim S(a \vee_1 b) \wedge S(a \vee_1 b)$.

Here $\wedge$, $\vee$, $\overline{}$ are the notations for the usual operations of conjunction, disjunction, and negation.

Theorem 1. If (Q_1) and (Q_2) are domains in n-dimensional space, defined respectively by the inequalities

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) \geqslant 0, \\ f_2(x_1, x_2, \dots, x_n) \geqslant 0, \end{aligned} \tag{1}$$

then the domain (Q), defined by the inequality

$$(x_1, x_2, \dots, x_n) = f_1 \wedge_1 f_2 \geqslant 0,$$

is the intersection (common part) of the domains (Q_1) and (Q_2).

It follows from property 8°.

Theorem 2. If (Q_1) and (Q_2) are domains defined by the inequalities (1), then the domain (Q), defined by the inequality

$$F(x_1, \dots, x_n) = f_1 \vee_1 f_2 \geqslant 0,$$

is the union of the domains (Q_1) and (Q_2).

It follows from 9°.

Fig. 1

Fig. 1

Figure 1: Fig. 1

Example 1. Let (Q_1) and (Q_2) be domains in the plane xOy, defined respectively by the inequalities $1-x^2 \geqslant 0$, $1-y^2 \geqslant 0$. Then the domain (Q), defined by the inequality

$$F(x,y) \equiv (1-x^2) \wedge_1 (1-y^2) = 2-x^2-y^2-\sqrt{(1-x^2)^2+(1-y^2)^2} \geqslant 0$$

is the interior of the square (Fig. 1a).

The equation

$$F(x,y) \equiv 2-x^2-y^2-\sqrt{(1-x^2)^2+(1-y^2)^2} = 0$$

is the equation of the square. Note that this equation is satisfied by the coordinates only of those points which lie on the sides of the square (but not on their extensions).

The domain defined by the inequality

$$F(x,y) \equiv (1-x^2) \vee_1 (1-y^2) = 2-x^2-y^2+\sqrt{(1-x^2)^2+(1-y^2)^2} \geqslant 0,$$

is shown in Fig. 1b.

Example 2. Let (Q_1), (Q_2), and (Q_3) be domains defined respectively by the inequalities $x \geqslant 0$, $y \geqslant 0$, $x+y \leqslant 1$.

The domain Q, defined by the inequality

$$F(x,y) \equiv (1-x-y) \wedge_1 (x \wedge_1 y) =$$

$$= 1-x-y+x+y-\sqrt{x^2+y^2}-\sqrt{(1-x-y)^2+\left(x+y-\sqrt{x^2+y^2}\right)^2} \geqslant 0,$$

is the interior of the triangle shown in Fig. 1c, and the equation

$$(1-x-y) \wedge_1 (x \wedge_1 y) = 0$$

is the equation of the triangle AOB.

§ 2. **Modified R-operations.** In what follows it proves convenient to use the following operations:

$$a \wedge_2 b = a - \sqrt{a^2 + b^2},$$

$$a \vee_2 b = a + \sqrt{a^2 + b^2}.$$

Properties of the modified operations.

1°. $a \wedge_2 b = 0 \sim S(a) \wedge b = 0$.

2°. $a \vee_2 b = 0 \sim \overline{S}(a) \wedge b = 0$.

Theorem 3. Let (Q) be a domain defined by the inequality

$$f(x_1, x_2, \dots, x_n) \geqslant 0,$$

and let L be a hypersurface defined by the equation

$$\varphi(x_1, \dots, x_n) = 0.$$

Then the equation

$$E(x_1, \dots, x_n) \equiv f \wedge_2 \varphi = 0$$

is satisfied by the coordinates of those and only those points of the n-dimensional space which belong to the domain (Q) and to the hypersurface L simultaneously. (At all other points the function F is negative.)

This follows from property 1° of the present section.

Example 3. Let (Q) be the strip $1 - x^2 \geq 0$, and let L be the straight line $y - x = 0$. Then

$$(1 - x^2) \wedge_2 (y - x) = 1 - x^2 - \sqrt{(1 - x^2)^2 + (y - x)^2} = 0$$

is the equation of the segment AB (Fig. 1 e).

§ 3. **Equation of a drawing.**

Theorem 4. *Let an arbitrary drawing be placed on the plane xOy, composed of n pieces of lines whose equations have the form $y = f(x)$ or $x = \varphi(y)$. Then one can write an equation $F(x, y) = 0$ that will be satisfied by the coordinates of those and only those points which belong to the given drawing.*

Proof. First consider the case when on the plane xOy there is drawn an arc AB of a curve L, having the equation $y = f(x)$ (Fig. 2 a). The vertical strip

Fig. 2

Figure 2: Fig. 2

between the points a and b is determined by the inequalities $a \leq x \leq b$, or $(b-x)(x-a) \geq 0$. On the basis of Theorem 3, the equation

$$F(x, y) \equiv (b-x)(x-a) \wedge_2 [y - f(x)] \geq 0$$

is the equation of the arc AB. At all other points of the plane xOy the function $F(x, y)$ is negative. Analogously one can write the equation of any arc of a curve defined by an equation of the form $x = \varphi(y)$.

Fig. 2

Now consider the general case, when on the plane xOy there is drawn a drawing composed of n pieces of lines of the indicated type. Write the equations of each of these pieces:

$$F_1(x, y) = 0; \quad F_2(x, y) = 0; \; \dots; \quad F_n(x, y) = 0$$

and form the equation

$$F(x, y) \equiv F_1 \cdot F_2 \cdot \dots \cdot F_n = 0. \tag{2}$$

If a point $M(x_0, y_0)$ is a point of the drawing, then at least one of the factors in formula (2) becomes zero, and, consequently, $F(x_0, y_0) = 0$. If this point does not belong to the drawing, then at this point all factors are negative, and, consequently, $F(x_0, y_0) \neq 0$. Therefore (2) is the equation of the drawing.

Let us note that if the functions $y = f(x)$ and $x = \varphi(y)$ are continuous, then the function $F(x, y)$ is of constant sign, namely, if $n = 2k$, then $F(x, y) \geq 0$; if $n = 2k + 1$, then $F(x, y) \leq 0$; in this case equality is attained only at points belonging to the drawing.

The equation $F(x, y) = c$ is the equation of a family of lines, to the number of which (for $c = 0$) the given drawing belongs.

All the arguments given above for the two-dimensional case are easily carried over also to the case of an n-dimensional space.

Example 4. Since the axes Ox and Oy intersect the domain $(1-x^2)(1-y^2) \geq 0$ only in the segments $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, respectively, the equation

$$F(x, y) = (1-x^2)(1-y^2) - \sqrt{(1-x^2)^2(1-y^2)^2 + x^2y^2}$$

is the equation of the cross $ABCD$ (Fig. 2 b).

Received
11 I 1963