MATHEMATICS

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1963

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Abstract

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MATHEMATICS

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CONDITIONS FOR PRESERVATION OF THE SIGN OF THE GREEN' S FUNCTION OF A STURM-LIOUVILLE PROBLEM

(Presented by Academician I. M. Vinogradov on 29 IX 1962)

Theorems on integral and differential inequalities serve as a rich source of various estimates for solutions of equations ([1–3]). In the works of the Izhevsk mathematical seminar (see, for example, ([3–5])) such theorems were obtained also for ordinary differential equations with multipoint Vallée-Poussin boundary conditions ([6]). It is natural to begin the investigation of the question of differential inequalities under more complicated boundary conditions with consideration of a second-order equation, which was suggested to me at the Izhevsk seminar by N. V. Azbelev and Z. B. Tsalyuk.

Consider the problem

$$\mathcal{L}[y] \equiv y'' + p(x)y' + q(x)y = f(x), \tag{1}$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = \beta_1 y(b) + \beta_2 y'(b) = 0 \tag{2}$$

with continuous p, q, and f.

If $G(x, s)$ is the Green' s function of problem (1)–(2), u is the solution of this problem, and z is some function satisfying the conditions (2), then, provided the sign of the Green' s function and of the residual $\varphi = \mathcal{L}[z] - f$ is preserved, the difference

$$z - u = \int_a^b G(x, s)\varphi(s)\, ds$$

also preserves its sign. Thus, Chaplygin' s theorem on a differential inequality for problem (1)–(2) is valid when the sign of the Green' s function is preserved. Below we give conditions for preservation of the sign of the function $G(x, s)$.

Lemma 1. Let $G(x, s)$ exist. In order that $G(x, s)$ preserve its sign in the square $x, s \in (a, b)$, it is necessary and sufficient that there exist a pair of functions $z(x)$ and $w(x)$, twice continuously differentiable on $[a, b]$, such that:

a) $z(a) = -\alpha_2, \quad z'(a) \leq \alpha_1 \ (z'(a) \geq \alpha_1), \quad z(x) > 0 \ (z(x) < 0), \quad z(x)\mathcal{L}[z(x)] \leq 0$ for $x \in (a, b)$;

b) $w(b) = \beta_2, \quad w'(b) \leq -\beta_1 \ (w'(b) \geq -\beta_1), \quad w(x) > 0 \ (w(x) < 0), \quad w(x)\mathcal{L}[w(x)] \leq 0$ for $x \in (a, b)$.

Here one cannot omit the condition of existence of $G(x, s)$, as is shown by the example: $y'' = 0,\ y(0) - y'(0) = y(b) - 2y'(b) = 0$. The functions $z(x) = x + 1$ and $w(x) = b - 2 - x$ satisfy the conditions of the lemma for any $b \in (0, 2)$. However, $G(x, s)$ does not exist if $b = 1$, and in the square $x, s \in (0, b)$ we have: $G(x, s) < 0$ if $b \in (0, 1)$, and $G(x, s) > 0$ if $b \in (1, 2)$.

Using this lemma, one can prove the following assertions on the existence and sign of the Green' s function.

Theorem 1. If $\beta_2 = 0$ ($\alpha_2 = 0$), then the existence of only one function $z(x)$ ($w(x)$), satisfying condition a) (b)) of Lemma 1, guarantees the existence of the function $G(x, s)$ and is a necessary and sufficient condition for the inequality $G(x, s) < 0$ to hold in the square $x, s \in (a, b)$.

It is easy to see that the assertions of Vallée-Poussin ([6]) on the existence of the Green' s function and of R. Bellman ([7]) on the sign of this function, which considered the case $\alpha_2 = \beta_2 = 0$, are special cases of the theorem given here.

Theorem 2. If $\alpha_2\beta_2 \neq 0$, then, for the existence of the function $G(x, s)$ and of the inequality $G(x, s) < 0$ in the square $x, s \in (a, b)$, it is necessary and sufficient that there exist a pair of twice continuously differentiable functions $z(x)$ and $w(x)$ on $[a, b]$ such that:

a) $z(a) = -\alpha_2, \quad z'(a) = \alpha_1, \quad z(x) \neq 0, \quad z(x)\mathcal{L}[z(x)] < 0$, for $x \in (a, b)$;

b) $w(b) = \beta_2, \quad w'(b) = -\beta_1, \quad w(x) \neq 0, \quad w(x)\mathcal{L}[w(x)] < 0$, for $x \in (a, b)$;

c) either $\beta_1 z(b) + \beta_2 z'(b) = \delta_1 \neq 0,\ \operatorname{sgn}\delta_1 = -\operatorname{sgn}(\alpha_2\beta_2)$, or $\alpha_1 w(a) + \alpha_2 w'(a) = \delta_2 \neq 0,\ \operatorname{sgn}\delta_2 = -\operatorname{sgn}(\alpha_2\beta_2)$.

Following the general scheme formulated as Lemma 3 in paper (8), one may take, as the functions $z(x)$ and $w(x)$ entering the conditions of the preceding assertions, solutions of certain auxiliary comparison equations. This leads to various comparison theorems. For example:

Theorem 3. Suppose two problems are given:

$$v'' - g_1(x)v = 0, \tag{3}$$

$$\alpha_1 v(a) + \alpha_2 v'(a) = v(b) = 0 \quad (v(a) = \beta_1 v(b) + \beta_2 v'(b) = 0); \tag{4}$$

$$y'' - g(x)y = 0, \tag{5}$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = y(b) = 0 \quad (y(a) = \beta_1 y(b) + \beta_2 y'(b) = 0). \tag{6}$$

If $g(x) \geq g_1(x)$ for $x \in [a, b]$ and the Green' s function $G_1(x, s)$ of problem (3)–(4) exists, with $G_1(x, s) < 0$ in the square $x, s \in (a, b)$, then there exists the Green' s function $G(x, s)$ of problem (5)–(6), and $G(x, s) < 0$ in the square $x, s \in (a, b)$.

In the case of a many-point boundary-value problem with Vallée-Poussin conditions (6), for any equation of order n there exists such a "subcritical" interval that, when the boundary conditions are posed in this interval, the Green' s function exists for the problem under consideration and Chaplygin' s theorem is valid ([3–5]). In the general Sturm-Liouville problem (1)–(2), the existence of a "subcritical" interval requires the fulfillment of additional conditions, which are given by

Lemma 2. For each operation $\mathcal{L}[y]$ there exists an interval $[\alpha, \rho)$ such that, for any $a, b \in [\alpha, \rho)$, $a < b$, the Green' s function $G(x, s)$ of problem (1)–(2) exists and preserves its sign for $x, s \in (a, b)$ if and only if either

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \neq 0,$$

or $\alpha_2 = \beta_2 = 0$. Moreover, we have:

a) if $\alpha_2 \beta_2 = 0$, then $G(x, s) < 0$ in the square $x, s \in (a, b)$;

b) if $\alpha_2 \beta_2 \neq 0$, then

$$\operatorname*{sgn}_{x,s \in (a,b) \subset [\alpha, \rho)} G(x, s) = \operatorname{sgn} \frac{\alpha_2 \beta_2}{\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}}.$$

The first three theorems concerned the case $G(x, s) < 0$. When $G(x, s) > 0$, one may use the following assertion:

Theorem 4. Suppose three problems are given

$$v'' - g_1(x) v = 0, \tag{7}$$

$$\alpha_1 v(a) + \alpha_2 v'(a) = v'(b) = 0 \quad (v'(a) = \beta_1 v(b) + \beta_2 v'(b) = 0); \tag{8}$$

$$u'' - g_2(x) u = 0, \tag{9}$$

$$\alpha_1 u(a) + \alpha_2 u'(a) = u'(b) = 0 \quad (u'(a) = \beta_1 u(b) + \beta_2 u'(b) = 0); \tag{10}$$

$$y'' - g(x)y = 0, \tag{11}$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = y'(b) = 0 \quad (y'(a) = \beta_1 y(b) + \beta_2 y'(b) = 0). \tag{12}$$

If $g_1 \le g \le g_2$ on $[a, b]$ and the Green' s functions of problems (7)–(8) and (9)–(10) exist and are positive in the square $x, s \in (a, b)$, then the Green' s function of problem (11)–(12) exists and is positive in the square $x, s \in (a, b)$.

For the proof of the nonlinear theorem given below, the following is used.

Lemma 3. *If the Green' s function of problem (1)–(2) is positive or negative in the square $x, s \in (a, b)$, and $c \in [a, b]$, then the Green' s function of the problem $\mathcal{L}[y] = 0$*

$$\alpha_1 y(a) + \alpha_2 y'(a) = y(c) = 0$$

$$(y(c) = \beta_1 y(b) + \beta_2 y'(b) = 0)$$

is negative in the square $x, s \in (a, c)$ $(x, s \in (c, b))$.

Using the ideas of papers ([5,8]) and the linear results considered above, one can obtain an assertion on a differential inequality and the existence and uniqueness theorems following from it for the nonlinear problem

$$N[y] \equiv y'' - f(x, y) = 0, \tag{13}$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = A, \qquad \beta_1 y(b) + \beta_2 y'(b) = B. \tag{14}$$

We assume that $f(x, y)$ is continuous and satisfies a Lipschitz condition in y in the domain $R : a \le x \le b,\ a_1 < y < b_1$. Thus, for $N[y]$ the conditions L_1 and L_2 of N. V. Azbelev ([3,8]) are simultaneously fulfilled; these consist in the existence of such linear operations $\mathcal{L}_j[y] \equiv y'' - p_j(x)y,\ j = 1, 2$, that for any ordered pair of functions $y_1 \ge y_2$ we have:

$$\mathcal{L}_1[y_1 - y_2] \ge N[y_1] - N[y_2] \ge \mathcal{L}_2[y_1 - y_2].$$

Denote by $G_j(x, s)$ the Green' s function for the equation $\mathcal{L}_j[y] = 0$ with conditions (2).

Theorem 5. *Let $G_1(x, s) > 0$ $(G_2(x, s) < 0)$ in the square $x, s \in (a, b)$. Suppose, further, that a pair of twice continuously differentiable functions $z(x)$*

and $v(x)$ on $[a,b]$, whose graphs lie in the domain R, satisfies the boundary conditions (14) and, for $x \in [a,b]$, the inequalities $N[z] \geq 0$, $N[v] \leq 0$. Then: a) there exists a solution u of problem (13)–(14); b) this solution can be obtained as the limit of monotonically convergent approximations of Chaplygin type, defined by the rule

$$u_{i+1} = \omega + \int_a^b G_j(x,s)\{\mathcal{L}_j[u_i(s)] - N[u_i(s)]\}\,ds, \qquad j = 1 \ (j = 2),$$

*where $u_0 = z$, or $u_0 = v$, and ω is the solution of the equation $\mathcal{L}_j[y] = 0$ satisfying the boundary conditions (14); c) if $G_1(x,s) < 0$ and there exists $G_2(x,s)$,** then the solution u is unique, and moreover $z \leq u \leq v$ for $x \in [a,b]$.*

It is easy to see that the results of papers ([9,10]) follow from Theorem 5.

In conclusion we note that A. L. Teptin and N. N. Yuberev recently reported, at the Izhevsk seminar, results extending the linear theorems given above to the case of equations in finite differences. These results make it possible, using a limiting transition from a difference equation to a differential one and the lemma of paper ([11]), to obtain additions to Theorem 5 in the case of an equation of the form $y'' = f(x, y, y')$.

I express my gratitude to the members of the Izhevsk seminar for their interest in the work and for valuable comments during the discussion of the results.

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Received
26 IX 1962

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* By virtue of the definition of the conditions L_1, L_2 and of a comparison theorem analogous to Theorem 3, one can be convinced that in this case $G_2(x, s) < 0$.

Note: Figure translations are in progress. See original paper for figures.