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Abstract

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ON THE OPTIMALITY OF VORONOI' S PRINCIPAL LATTICE OF THE FIRST TYPE AMONG LATTICES OF THE FIRST TYPE OF ANY NUMBER OF DIMENSIONS

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Let in the n-dimensional Euclidean space E^n there be given some lattice Γ. To the lattice Γ there corresponds a number R_Γ such that balls of radius R_Γ, with centers at the points of this lattice, cover the entire space E^n, while balls of smaller radius do not cover it. By a lattice covering of the space E^n we shall mean only a covering of the space E^n by balls of radius $R = R_\Gamma$ with centers at the points of the lattice Γ.

The density of a lattice covering is defined as the ratio of the volume of an n-dimensional ball of radius R_Γ to the volume of the fundamental parallelepiped of the lattice Γ. Obviously, the density is invariant under similarity.

The lattice of the n-dimensional space E^n constructed on the regular Selling frame, i.e., on a Selling frame with equal scalar products of its vectors, will be denoted by Γ_1^n and will be called the **principal lattice of the first type of Voronoi** (^{1}).

The main result of this article is the following theorem.

Theorem. *To the lattice Γ_1^n there corresponds the least value of the density of lattice coverings of the space E^n among lattices of the first type of Voronoi.*

1. Let us denote by Γ a lattice of the space E^n; by D, its Dirichlet domain; and the distances from the center of the domain D to the vertices of the domain D by R_λ ($\lambda = 1, 2, \dots, N$, where N is the number of vertices of the domain D), henceforth called the **radii of the Dirichlet domain**. Obviously, the radius R_Γ is the greatest among the radii R_λ, or, what is the same, the greatest of the radii of the balls (L) circumscribed about the polyhedra L of the star of the decomposition $\{L\}$ corresponding to the domain D (^{2}). The problem reduces to finding the least value of the expression $\max_\lambda(R_\lambda^n/V)$, or, what is the same,

$$\max_\lambda \left(R_\lambda^2/\sqrt[n]{V^2}\right),$$

where by V is denoted the volume of the fundamental parallelepiped of the lattice Γ.

The proof of the theorem is based on B. N. Delone' s idea of seeking the least value of the expression

$$M=\sum_{\lambda=1}^{N}\left(R_{\lambda}^{2}/\sqrt[n]{V^{2}}\right)$$

and on the following lemma:

Lemma. *If in the space E^n there is given a system of q points $\{A_i\}$ $(i=1,2,\dots,q)$, then the sum of the squares of the distances from any point C to the points of the system $\{A_i\}$ is equal to the sum of the squares of the distances from the center of gravity O of the system of points $\{A_i\}$ to the points of this system, added to the product of the number of points of the system by the square of the distance between O and C.*

Indeed, let $\overrightarrow{CA_i}$ be the vector going from the point C to the point A_i, $\overrightarrow{OA_i}$ the vector going from the center of gravity O to the point A_i, and $\overrightarrow{CO}$ the vector from the point C to the point O.

Then

$$\overrightarrow{CA_i}=\overrightarrow{OA_i}+\overrightarrow{CO},$$

$$\overrightarrow{CA_i}^{2}=\overrightarrow{OA_i}^{2}+\overrightarrow{CO}^{2}+2\overrightarrow{CO}\cdot\overrightarrow{OA_i},$$

whence we have that

$$\sum_{i=1}^{q}\overrightarrow{CA_i}^{2}=\sum_{i=1}^{q}\overrightarrow{OA_i}^{2}+q\cdot\overrightarrow{CO}^{2},$$

since $\sum_{i=1}^{q}\overrightarrow{OA_i}=0$. We note that this lemma is equivalent to the theorem of mechanics on the moment of inertia of a system of point masses with respect to an axis.

2. Derivation of a formula. Let $(\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_{n+1})$ be a reduced Selling frame of some lattice Γ of the first type. In what follows, by the words "the lattice Γ"we shall mean a lattice Γ of the first type (3) and, in addition, shall assume that the lattice Γ is primitive, since nonprimitive lattices are limiting cases of primitive ones (2). A **snake** $\langle\mathbf{a}_{k_0},\mathbf{a}_{k_1},\dots,\mathbf{a}_{k_n}\rangle$ of the vectors of the reduced Selling frame $(\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_{n+1})$ is called a closed broken line with links $\mathbf{a}_{k_0},\mathbf{a}_{k_1},\dots,\mathbf{a}_{k_n}$, where $k_0,k_1,\dots,k_n$ is some permutation of the numbers $1,2,\dots,n+1$ (4). From the works of G. F. Voronoi (3) it follows that the simplexes (the convex hulls of snakes) constructed on all possible snakes of the vectors of the reduced Selling frame of the lattice Γ, with the beginning of the first link at the point A_0, are simplexes L and together form the star of the point A_0 in the partition $\{L\}$.

Consider some simplex L_k of the star of the point A_0 of the lattice of the partition $\{L\}$; let O_k and C_k be, respectively, the center of gravity and the center of the circumscribed sphere of the simplex L_k, and let $A_0, A_1, \dots, A_n$ be its vertices.

Introduce the notation: $\overrightarrow{O_{kA}i} = \mathbf{b}_{ki}$, $\overrightarrow{C_{kA}i} = \mathbf{R}_{ki}$, $|\mathbf{R}_{ki}| = R_k$, $\overrightarrow{O_{kC}k} = \mathbf{c}_k$. Then, by the lemma,

$$R_k^2 = \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{b}_{ki}^2 + \mathbf{c}_k^2,$$

and the sum of the squares of the radii of the domain D will have the form

$$\sum_{k=1}^{(n+1)!} R_k^2 = \frac{1}{n+1} \sum_{k=1}^{(n+1)!} \sum_{i=0}^{n} \mathbf{b}_{ki}^2 + \sum_{k=1}^{(n+1)!} \mathbf{c}_k^2.$$

Express

$$\sum_{k=1}^{(n+1)!} \sum_{i=0}^{n} \mathbf{b}_{ki}^2$$

in terms of the sum of the squares of the vectors of the reduced Selling frame of the lattice Γ. Let the simplex L_k be constructed on the snake $\langle \mathbf{a}_{k_0}, \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_n} \rangle$. Then the coordinates of its vertices $A_i, A_{i+1}, \dots, A_{n-i}$ $(i = 0, 1, \dots, n)$ with respect to the frame $\mathbf{a}_{k_i}, \mathbf{a}_{k_{i+1}}, \dots, \mathbf{a}_{k_{n-i}}$ of n vectors of the reduced Selling frame will be $(0, 0, \dots, 0)$, $(1, 0, 0, \dots, 0)$, $(1, 1, 0, \dots, 0), \dots, (1, 1, \dots, 1)$, and the vector $\mathbf{b}_{ik}$, going from the vertex A_i to the center of gravity O_k of the simplex L_k, is expressed through the vectors of the snake $\mathbf{a}_{k_0}, \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_n}$ as follows:

$$\mathbf{b}_{ik} = \frac{n\mathbf{a}_{k_i} + (n-1)\mathbf{a}_{k_{i+1}} + \dots + \mathbf{a}_{k_{i+n-1}}}{n+1},$$

where $i = 0, 1, \dots, n$, $\mathbf{a}_{k_{i+n-1}} = \mathbf{a}_{k_{i-2}}$ for $i = 2, 3, \dots, n$, whence for

$$\sum_{k=1}^{(n+1)!} \sum_{i=0}^{n} \mathbf{b}_{ik}^2$$

after some elementary computations, taking into account the rela-

using

$$2\sum_{i=1}^{n} a_i a_{i+1} = -\sum_{j=1}^{n+1} a_j^2,$$

which follows from the fact that

$$\sum_{j=1}^{n+1} a_j = 0,$$

we find the expression

$$\sum_{k=1}^{(n+1)!} \sum_{i=0}^{n} b_{ik}^2 = \frac{(n+1)!}{12}(n+2)\sum_{j=1}^{n+1} a_j^2,$$

and, consequently, the sum of the squares of the radii of the Dirichlet domain of the lattice Γ takes the form:

$$\sum_{k=1}^{(n+1)!} R_k^2 = \frac{(n+1)!}{12} \cdot \frac{n+2}{n+1} \sum_{j=1}^{n+1} a_j^2 + \sum_{k=1}^{(n+1)!} c_k^2. \tag{1}$$

3. Proof of the theorem. Let $g_{ms} = a_m a_s$ $(m < s,\ m = 1, 2, \dots, n,\ s = 2, 3, \dots, n+1)$ be the Selling parameters of the lattice Γ. We shall show that on the surface $\Delta = (n+1)^{n-1}$ in the $n(n+1)/2$-dimensional space of the Selling parameters g_{ms} of lattices, where Δ is the square of the volume of the fundamental parallelepiped of the lattice Γ, the sum of the squares of the radii of the Dirichlet domain attains its least value at the point $P = (-1, -1, \dots, -1)$, which corresponds to the lattice Γ_1^n. Let us examine the expression $B = \sum_{j=1}^{n+1} a_j^2$, which enters the first term of formula (1). In the space of the parameters g_{ms},

$$\sum_{j=1}^{n+1} a_j^2 = -2(g_{12} + g_{13} + \dots + g_{n\,n+1}).$$

Therefore $B = \sum_{j=1}^{n+1} a_j^2$ in the space of these parameters is the equation of a hyperplane Q. The discriminant surface $\Delta = \text{const}$, as Minkowski showed (^{5}), is convex toward the origin. It is obvious that for a certain value of B the hyperplane Q is tangent to the surface $\Delta = (n+1)^{n-1}$ at the point $P = (-1, -1, \dots, -1)$. Consequently, the function $\sum_{j=1}^{n+1} a_j^2$ on the surface $\Delta = (n+1)^{n-1}$ has at this point a unique minimum, which is also its least value. The expression $\sum_{k=1}^{(n+1)!} c_k^2 \geqslant 0$ at the point P is equal to zero, since for the simplexes L of the partition $\{L\}$ of the lattice Γ_1^n, as is easy to see, the centers of gravity coincide with the centers of the circumscribed spheres.

Consequently, the expression M of item 1 for the lattice Γ_1^n attains its least value. But since at the point P all R_k are equal, the largest of the R_k, for fixed Δ, also attains its least value at the point $P = (-1, -1, \dots, -1)$.

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Note: Figure translations are in progress. See original paper for figures.

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