MATHEMATICS

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Abstract

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DENSITY OF FINITE FUNCTIONS IN THE SPACE $L_p^{(m)}(E_n)$

In the present note we shall give a direct proof of the following theorem.

Theorem. In the space of classes of functions $L_p^{(m)}(E_n)$ $(p > 1)$, defined in the whole Euclidean space E_n with norm

$$\|f\|^p_{L_p^{(m)}(E_n)} = \int \cdots \int \left[\sum_{|\alpha|=m} (D^\alpha f)^2 \right]^{p/2} dx, \tag{1}$$

finite functions form a dense set. In other words, the space $L_p^{(m)}(E_n)$ coincides with the closure of the set of finite functions in the norm (1).

It is enough to restrict ourselves to establishing that, by means of finite functions in the metric $L_p^{(m)}(E_n)$, one can approximate such elements of this space which:

a) are infinitely differentiable everywhere in E_n;

b) have partial derivatives $D^\alpha f$ of any order α: $|\alpha| = m - k$, $0 \leq k < n/p$, tending to zero at infinity in almost all directions, and moreover in such a way that

$$\int \cdots \int |D^\alpha f|^p r^{n-pk-1} \, dr < \infty; \tag{2}$$

c) the function f vanishes identically for $r > 3$.

Let f_h be the mean function for f. For the space L_p [1]

$$\|f - f_h\| \leq A\delta(h), \tag{3}$$

where

$$\delta(h) \leq \max_{|\Delta y| \leq h} \|f(y) - f(y + \Delta y)\|_{L_p}. \tag{4}$$

Applying the inequality (4) to $D^\alpha f$ ($|\alpha| = m$), we see that $D^\alpha f_h$ approximates $D^\alpha f$, whence a) follows. The restriction b) is proved by complete induction from derivatives of higher order to derivatives of lower order. Let

$$\int_0^\infty \left[\sum_{|\alpha|=m} (D^\alpha f)^2\right]^{p/2} r^{\,n-1}\,dr = C^p(\gamma). \tag{5}$$

Then $C^p(\gamma)$ is an integrable function of the angle γ:

$$\int_\gamma C^p(\gamma)\,d\gamma = \|f\|^p_{L_p^{(m)}(E_n)}. \tag{6}$$

We shall show successively that for all derivatives of the order under consideration and for almost all γ the inequality

$$\int_i^\infty |D^\alpha f|^p r^{\,n-kp-1}\,dr < KC^p(\gamma), \qquad |\alpha| = m-k, \quad k < \frac{n}{p}, \tag{7}$$

holds.

where K is a constant independent of f. From formula (7) it will follow, among other things, that

$$\int_{r>l} \cdots \int |D^\alpha f|^p r^{-kp}\,dx < K\|f\|^p_{L_p^{(m)}(E_n)}. \tag{8}$$

For $k = 0$ formula (7) is, obviously, true. We express derivatives of order $m-k$ in terms of derivatives of order $m-k+1$ by integrating the total differential

$$D^\alpha f = C_\alpha(\gamma) - \int_x^\infty \frac{d}{ds}(D^\alpha f)\,ds \tag{9}$$

along paths l, part of which is a ray going to infinity from the origin of coordinates. For derivatives of order $m-k+1$ we regard formula (7) as proved. Formula (8) has meaning for almost all rays. For almost all γ the constant $C_\alpha(\gamma)$ will be one and the same.

If there were two sets of nonzero measure with different values of $C_\alpha(\gamma)$, then there would be a set of paths, depending on $(n-2)$ parameters, lying on the sphere and having nonzero measure, along which

$$\lim_{r=R} \int \frac{d}{ds}(D^\alpha f)\,ds = \lim R \int \frac{d}{ds}(D^\alpha f)\,ds_\gamma > \eta > 0. \tag{10}$$

Hence the integral

$$\int_\gamma \int_l^\infty |D^\alpha f|^p r^{n-kp-1}\, dr\, d\gamma,$$

would diverge, which contradicts (7), already established for $|\alpha| = m-k+1$. Correcting f by the corresponding polynomial of degree lower than $m-k+1$, we obtain, for the equivalent function, the formula

$$D^\alpha f = -\int_x^\infty \frac{d}{ds}(D^\alpha f)\, ds. \tag{11}$$

Hence, by applying Hardy' s inequality ((2), p. 302)

$$\int_0^\infty t^{-r}|F(t)|^p\, dt < \left(\frac{p}{|r-1|}\right)^p \int_0^\infty t^{-r}|tf(t)|^p\, dt, \tag{12}$$

where $r<1$ and $F(x)=\int_x^\infty f(t)\, dt$, we establish (7).

Functions satisfying a) and b) can always be decomposed into two summands, one of which is finite, while the other is equal to zero in the ball of radius $r=3$ about the origin. Hence it follows that it is sufficient to restrict oneself to functions satisfying c).

For functions satisfying a), b), c) and inequality (7), derivatives of any order, including orders not exceeding $m-n/p$, can be represented by the formula

$$D^\alpha f = \int_0^x \frac{d}{ds}(D^\alpha f)\, ds. \tag{13}$$

If n is not a multiple of p, then to the integrals (13) one can again apply Hardy' s inequality (12) for $r>1$ and $F(x)=\int_0^x f(t)\, dt$. This establishes the validity of

formulas (7) and (8) also for $n/p<k\leq m$. In the case $n=kp$, instead of formula (7) one can establish the cruder inequalities:

$$\int_l^\infty |D^\alpha f|^p r^{n-kp-1}(\ln r)^{-p}(\ln\ln r)^{-p}\, dr < KC^p(\gamma) \tag{14}$$

for almost all γ, and, consequently,

$$\int_{r>l}\cdots\int |D^\alpha f|^p r^{-kp}(\ln r)^{-r}(\ln\ln r)^{-p}\, dr < K\|f\|^p_{L_p^{(m)}}, \tag{15}$$

which also replaces formula (8), being valid for all derivatives of f of order not exceeding m.

Instead of (15), let us establish a stronger inequality, valid for almost all γ:

$$|D^\alpha f|^p \leq r^{kp-n}(\ln r)^{p-1} K^p C^p(\gamma), \qquad |\alpha| = m - \frac{n}{p}, \tag{16}$$

or

$$|D^\alpha f| < r^{k-n/p}(\ln r)^{1/p'} K C(\gamma). \tag{17}$$

Let the function $f(x)$ satisfy the condition

$$\int_l^\infty |f(x)|^p x^{p-1}\, dx = C^p < \infty, \tag{18}$$

and let

$$F(x) = \int_0^x f(t)\, dt.$$

Introduce a new variable y by the formula

$$y = \frac{1}{(p-1)(\ln x)^{p-1}}. \tag{19}$$

Denote

$$f(x) x^{(p-1)/p} \left(\frac{dx}{dy}\right)^{1/p} = \chi(y). \tag{20}$$

The nonincreasing rearrangement $\overline{\chi}(y)$ of the function $\chi(y)$, by virtue of (18), is estimated as

$$\overline{\chi}(y) \leq \frac{C}{y^{1/p}}. \tag{21}$$

On the other hand,

$$F(x) = \frac{1}{(p-1)^{p'}} \int_y^{1/(p-1)} \frac{\chi(z)}{z}\, dz \leq K \int_y^{1/(p-1)} \frac{\overline{\chi}(z-y)}{z}\, dz \leq \frac{KC}{y^{1/p}}. \tag{22}$$

Applying (22) to $D^\alpha f$, $|\alpha| = m - n/p$, we obtain

$$|D^\alpha f| \leq K C(\gamma)(\ln r)^{1/p'}, \qquad |\alpha| = m - \frac{n}{p}. \tag{23}$$

The last estimate gives us formula (16), and therefore also (15). Now putting

$$\psi_h(x) = \psi\left(\frac{\ln\ln r}{\ln|\ln h|}\right), \tag{24}$$

where $\psi(\xi)$ is continuous together with all derivatives,

$$\psi(\xi) = \begin{cases} 1, & \xi < 1/2, \\ 0, & \xi > 1, \end{cases} \tag{25}$$

we shall have

$$\frac{d^k}{dr^k}\psi_h \leqslant \frac{K}{\ln\ln r}\cdot\frac{1}{\ln r}\cdot\frac{1}{r^k}. \tag{26}$$

After this it is easy to establish the property

$$\lim_{h\to 0}\|f - f\psi_h\|_{L_p^{(m)}(E_n)} = 0. \tag{27}$$

For $n \not\equiv 0 \pmod p$, as the cut-off function one may take

$$\psi_h^*(r) = \psi(hr). \tag{28}$$

A simple example:

$$f(r) = \left[{}^1\!/_2\ \ln(r^2+h^2)\right]^{1/2-\varepsilon}, \quad h = 2, \quad p = 2, \quad m = 1,$$

shows that when n is a multiple of p, $\|f - f\psi_h\|$ does not tend to zero, and $\psi^*(r)$ is unsuitable as a cut-off function. From this same example it is seen that the exponent of $\ln r$ in formula (23) cannot be improved.

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REFERENCES

1. S. L. Sobolev, Some applications of functional analysis in mathematical physics, L., 1950.
2. G. Hardy, J. Littlewood, G. Polya, Inequalities, M., 1948.