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Abstract

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OPTIMAL MODES OF MOTION OF A POINT OF VARIABLE MASS IN A HOMOGENEOUS CENTRAL FIELD

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I. With the aid of L. S. Pontryagin' s maximum principle, the problem of optimal control of a point of variable mass in a central gravitational field is considered in the presence of a constraint on the reactive thrust $0 \leq P(t) \leq P_{\max}$. The exact solution of this problem for a central field, in contrast to a plane-parallel homogeneous field, is complicated by the absence of a solution of the system of adjoint equations. Below an approximate solution is obtained, based on the assumption that the motion takes place in a sufficiently narrow cylindrical (spherical) layer. The thickness of the layer Δr is assumed such that quantities of order $\Delta r/r_{\text{cp}}$, where r_{cp} is the distance from the center of attraction to the middle of the layer, may be neglected in comparison with unity. Under this assumption the acceleration of gravity $g(r)$ is constant in magnitude but variable in direction. In this connection, by analogy with a homogeneous plane-parallel field, the field under consideration may be called a homogeneous central field. The advantages of such a field model, as compared with the model of a field linearized in the neighborhood of the origin of the starting coordinate system ([1]), are obvious in the case when the flight range is comparable with r_{cp} or, still more, exceeds this quantity.

The equations of motion of a point of variable mass in a homogeneous-cylindrical field in the Cartesian coordinate system Oxy, with origin at the center of attraction, have the form:

$$\dot{u} = \frac{P}{m}\cos\varphi - \nu^2 x; \qquad \dot{v} = \frac{P}{m}\sin\varphi - \nu^2 y, \qquad \dot{x} = u; \qquad \dot{y} = v;$$

$$\dot{m} = -\frac{1}{c}P, \tag{1}$$

where φ is the inclination of the reactive force to the axis Ox; $c = \text{const}$ is the exhaust velocity; $\nu = \sqrt{g(r_{\text{cp}})/r_{\text{cp}}}$ is the angular velocity of the radius vector of a satellite moving in a circular orbit of radius r_{cp}. It should be noted that these equations are exact in the central field of "elastic" forces, in which $g(r) = \nu^2 r$.

In Newton' s central gravitational field, equations (1) approximately describe motion that does not leave the limits of a sufficiently narrow cylindrical (spherical) layer.

Let us pose the problem of determining the control functions $P(t)$ and $\varphi(t)$ ($0 \leq P(t) \leq P_{\max}$) so that, when the system passes from some initial position to some final position corresponding to $t = T$, the functional

$$S = C_u u(T) + C_v v(T) + C_x x(T) + C_y y(T) + C_m m(T),$$

where C_u, C_v, C_x, C_y, C_m are constants, has the greatest possible value. For equations (1), the function H has the form (see ([2]))

$$H = \frac{P}{m}\left(p_u \cos\varphi + p_v \sin\varphi - \frac{m}{c} p_m\right) - \nu^2 (p_u x + p_v y) + p_x u + p_y v, \tag{2}$$

where p_u, p_v, p_x, p_y, and p_m denote the adjoint variables corresponding to the variables u, v, x, y, and m.

The conditions for the minimum of this function, necessary for ensuring the maximum of the functional S, have the form

$$\sin\varphi = -\frac{p_v}{\rho}; \qquad \cos\varphi = -\frac{p_u}{\rho}; \qquad P = \begin{cases} P_{\max}, & \text{when } \vartheta > 0, \\ 0, & \text{when } \vartheta < 0, \end{cases} \tag{3}$$

where $\rho = \sqrt{p_u^2 + p_v^2}$; $\vartheta = \rho + m\frac{p_m}{c}$ is the switching function, and the adjoint variables are determined from the equations:

$$\dot{p}_u = -p_x; \quad \dot{p}_v = -p_y; \quad \dot{p}_x = \nu^2 p_u; \quad \dot{p}_y = \nu^2 p_v; \quad \dot{p}_m = -\frac{P\rho}{m^2}. \tag{4}$$

The boundary conditions for this system are found according to the rules ([2], [3]). Equalities (3) hold under the condition that $\vartheta \neq 0$. From equalities (3) it is seen that the character of the optimal control is determined by the behavior of the solutions of equations (4). In particular, the direction of the thrust in the coordinate system Oxy is parallel to the vector with coordinates $x = -p_u$ and $y = -p_v$.

In the case under consideration of a homogeneous central field, system (4) has the solution

$$p_u = p_u^0 \cos\nu t - \frac{p_x^0}{\nu}\sin\nu t,$$

$$p_v = p_v^0 \cos\nu t - \frac{p_y^0}{\nu}\sin\nu t. \tag{5}$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

The additional superscript 0 denotes the initial values of the variables.

It follows from (5) that the p-trajectory (the projection of the integral curve of system (4) onto the plane of the adjoint variables p_u and p_v) is a central ellipse. The limiting types of p-trajectories that should be singled out are the circle and segments of two coincident straight lines passing through the origin. Following (4), p-trajectories passing through the origin will be called special.

The existence of solution (5) makes it possible to clarify the structure of the optimal control in a homogeneous central field. The analysis is based on the method developed in the study of optimal control in a plane-parallel field ([4]).

Fig. 1

II. From equalities (3) and (5) it follows that the optimal program for the variation of the thrust inclination angle φ has the form

$$\varphi = \operatorname{Arc\,tg}\left(\frac{\nu p_v^0 \cos \nu t - p_y^0 \sin \nu t}{\nu p_u^0 \cos \nu t - p_x^0 \sin \nu t}\right). \tag{6}$$

The limiting cases of this program are the linear dependence $\varphi = \varphi_0 \pm \nu t$, which is obtained for a circular p-trajectory, and a discontinuous control law, realized in the case of special p-trajectories. In the latter case φ assumes a sequence of constant values φ_0, $\varphi_0 \pm \pi$, $\varphi_0 \pm 2\pi$, $\varphi_0 \pm 3\pi$, etc. The increments of φ occur, respectively, at the times t_1, $t_1 + \pi/\nu$, $t_1 + 2\pi/\nu$, etc., where t_1 is the time of the first discontinuity, and φ_0 is the initial value of φ. Characteristic types of optimal

control of the direction of thrust, together with the corresponding types of p-trajectories, are shown in Fig. 1. It should be noted that the variation of φ is monotonic.

The transition to the case of a plane-parallel field is effected by decreasing the time interval of motion under consideration. If in (6) the principal terms for small ν are retained, then under the arctangent sign one obtains the well-known fractional-linear dependence.

Fig. 2

III. The optimal program for regulating the magnitude of the thrust force is of boundary type (the thrust assumes either a maximum or a minimum

Fig. 3

Figure 3: Fig. 3

value), provided that $\vartheta \neq 0$. If $\vartheta \equiv 0$, then a singular control (3) is possible, in which the magnitude of the thrust force is regulated. From (1), (3), and (4) we have:

$$\dot{\vartheta} + \frac{P}{mc}\vartheta = \dot{\rho}. \tag{7}$$

Hence it is seen that singular control is possible if and only if

$$\rho = \sqrt{\rho_u^2 + \rho_v^2} = \text{const},$$

$$\vartheta_0 = \rho_0 + m_0\frac{\rho_m^0}{c} = 0, \tag{8}$$

where ϑ_0 is the value of the switching function at the initial instant. Thus, singular control is possible only in the case of circular p-trajectories. The case of realization of the optimal regime with $\rho =$ const was considered in (5), but there the second of the necessary conditions (8) for the existence of singular control was not indicated. At the same time, the first of conditions (8), as follows from equalities (3) and (7), is only the condition for preservation of the regime over the entire interval of variation of time, and is valid for any of the three types of control of the thrust magnitude indicated in (5).

Fig. 3

For $\vartheta \neq 0$, the character of the variation of the thrust magnitude is determined by the mutual arrangement of the curves $\rho(t)$ and $z = -mp_m/c$. From (1), (4), and (5) it follows that

$\rho(t)$ is a periodic function, while $z(t)$ is a monotonically increasing function of time (see Fig. 2).

Taking this into account, it is not difficult to establish that, for $\vartheta^0 \neq 0$, five types of control regimes for the thrust magnitude are possible (Fig. 3):

a. $P(t) \equiv P_{\max}$—an active segment over the entire time interval under consideration $0 \leq t \leq T$.

b. $P(0) = P_{\max}$, $P(T) = P_{\max}$—an active segment at the beginning and an active segment at the end of the motion.

c. $P(0) = P_{\max}$, $P(T) = 0$—an active segment at the beginning of the motion and a passive segment at the end.

d. $P(0) = 0,\ P(T) = P_{\max}$—a passive segment at the beginning and an active segment at the end.

e. $P(0) = 0,\ P(T) = 0$—the motion begins and ends with passive segments. The active segments are located inside the interval $0 \leq t \leq T$.

The total number of active segments depends on the number of complete cycles described by the point along the p-trajectory and increases without bound as $T \to \infty$, while the duration Δt_k of the active segment with number k ($k \geq 3$) decreases as k increases. In this case the boundaries of the passive segments (except, perhaps, the first and last in the composition of the optimal trajectory) —similarly to the case of the plane-parallel field ([4])$_k$—are symmetric with respect to the instants of time corresponding to the minimum values of $\rho(t)$ and forming a sequence of points t_{0k} ($k = 1, \dots, n$) with period π/ν.

All the properties of the optimal control formulated above carry over to the case of spatial motion. This is a consequence of the fact that in the spatial case, just as in the planar case, the p-trajectory is a plane ellipse.

A homogeneous central field is a generalization of a plane-parallel field. However, the periodic character of the p-trajectory leads to a number of qualitative features, among them: the possibility of realizing a large number of active segments on optimal trajectories of long duration; the possibility of a singular control $0 < P(t) < P_{\max}$; and, finally, a greater variety of optimal-control regimes—the regime admissible here with an active segment inside the interval of motion has no analogue in the plane-parallel field ([6]).

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Note: Figure translations are in progress. See original paper for figures.