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Abstract

Full Text

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TEST SPACES

(Presented by Academician P. S. Aleksandrov on 27 IV 1963)

We use the following notation: $\dim_G X$ is the cohomological dimension of the compactum X with respect to the coefficient group G ; Z is the infinite cyclic group; Z_p is the cyclic group of order p ; Q is the additive group of rational numbers; R_p is the additive group of those rational numbers which, when written as an irreducible fraction, have denominator relatively prime to p ; Q_p is the additive group of rational numbers of the form n/p^α , reduced modulo 1. By definition, the system of groups σ consists of the groups Z_p, Q_p, R_p, Q , and the index p runs through all prime values. All compacta in what follows are assumed to be finite-dimensional.

M. F. Bokshtein ⁽²⁾ indicated a method for computing the cohomological dimension of a compactum with respect to an arbitrary coefficient group, if its dimensions with respect to the groups of the system σ are known.

We consider functions defined on the set σ and taking values in the set of nonnegative integers. A function D is called a dimension function if, for every prime number p , it satisfies the following conditions:

1. $D(Q_p) \leq D(Z_p) \leq D(Q_p) + 1$.
 2. $D(R_p) \leq \max(D(Q), D(Q_p) + 1)$.
 3. $D(Q_p) \leq \max(D(Q), D(R_p) - 1)$.
 4. $D(Z_p) \leq D(R_p)$.
 5. $D(Q) \leq D(R_p)$.
 6. If $D(G_0) = 0$ for some group $G_0 \in \sigma$, then $D(G) = 0$ for every group $G \in \sigma$.
- (1)

The number $\dim D = \max_{G \in \sigma} D(G)$ is called the dimension of the function D . The function

$$D(G) = \dim_G X$$

is called the dimension function of the compactum X .

The dimension function of an arbitrary n -dimensional compactum X is an n -dimensional dimension function. This assertion is a combination of known re-

sults of P. S. Aleksandrov and M. F. Bokshtein. The following theorem gives a partial answer to the question of whether the converse assertion is true.

Theorem 1. *An arbitrary n -dimensional dimension function D satisfying the condition $D(G) \geq [n/2]$ for every group $G \in \sigma$ is the dimension function of some n -dimensional compactum.*

Corollary 1. *An arbitrary n -dimensional dimension function for $n \leq 3$ is the dimension function of some n -dimensional compactum.*

Corollary 2. *For each group $G \in \sigma$ and each integer $n \geq 1$ there exists a $2n$ -dimensional compactum X_G^n , whose dimensions are determined by the following table:*

	$\dim R_p$	$\dim Z_p$	$\dim Q_p$	$\dim R_q$	$\dim Z_q$	$\dim Q_q$	$\dim Q$
$X_{R_p}^n$	$2n$	$2n$	$2n$	$2n$	n	n	$2n$
$X_{Z_p}^n$	$2n$	$2n$	$2n - 1$	n	n	n	n
$X_{Q_p}^n$	$2n$	$2n - 1$	$2n - 1$	n	n	n	n
X_Q^n	$2n$	n	n	$2n$	n	n	$2n$

(In this table q is an arbitrary prime number distinct from p .)

Examples of compacta having the dimensions indicated in the table are the following: $X_{Z_p}^n$ —the n -th topological power of the two-dimensional compactum of L. S. Pontryagin ⁽³⁾, $X_{R_p}^n$ and X_Q^n —the n -th topological powers of the two-dimensional compacta of Kodama ⁽⁶⁾, $X_{Q_p}^1$ —the two-dimensional compactum of V. G. Boltyanskii. The compacta $X_{Q_p}^n$ for $n > 1$ cannot be obtained as products of two-dimensional compacta. They can be constructed by modifying the construction indicated in the note ⁽⁸⁾. The compacta X_G^n are called test spaces. For $n = 1$ they were considered by Kodama ⁽⁶⁾.

Lemma. For an arbitrary compactum X and any group $G \in \sigma$, the equality

$$\dim(X \times X_G^n) = \max(\dim X + n, \dim_G X + \dim X_G^n)$$

holds.

Proof. The lemma is proved by direct computations using Bockstein's formulas, which express the dimensions of a product of compacta in terms of the dimensions of the factors *.

Theorem 2. If X is an arbitrary finite-dimensional compactum, G is a group from the system of groups σ , and d is a nonnegative integer, then the inequality $\dim_G X \geq \dim X - d$ is equivalent to the inequality

$$\dim(X \times X_G^{d+1}) \geq \dim X + \dim X_G^{d+1} - d.$$

Proof. Let $\dim_G X \geq \dim X - d$. Then, by the preceding lemma:

$$\dim(X \times X_G^{d+1}) = \max(\dim X + d + 1, \dim_G X + \dim X_G^{d+1}) \geq \dim X + \dim X_G^{d+1} - d.$$

Conversely, suppose

$$\dim(X \times X_G^{d+1}) = \dim X + \dim X_G^{d+1} - d.$$

Then

$$\dim(X \times X_G^{d+1}) = \dim_G X + \dim X_G^{d+1}$$

and $\dim_G X \geq \dim X - d$. The theorem is proved.

For $d = 0$ this theorem becomes Kodama's theorem ⁽⁶⁾.

Theorem 2 makes it possible to compute the cohomological dimensions of a compactum from the Urysohn dimensions of its products with test spaces. Therefore, knowing the dimensions \dim of the products of compacta X and Y with test spaces, one can determine the dimension $\dim(X \times Y)$. Often, in order to solve problems connected with computing the dimension of a product, it is sufficient to know the dimensions of the products not with all test spaces, but only with some of them. For example, in order to establish whether, for given compacta X and Y and an integer d , the inequality

$$\dim(X \times Y) \geq \dim X + \dim Y - d$$

holds, it is sufficient to know the dimensions of the products of the compacta X and Y with the compacta X_G^{d+1} . (For $d = 0$ this result was established by Kodama ⁽⁵⁾.) Other examples are contained in the following theorems.

Generalizing P. S. Aleksandrov's definition of dimensional completeness of a compactum, let us call the dimensional defect of a compactum X the integer $d(X)$ such that there exists a compactum Y for which $\dim(X \times Y) = \dim X + \dim Y - d(X)$

$$\begin{aligned} \dim_{Z_p}(X \times Y) &= \dim_{Z_p} X + \dim_{Z_p} Y, \\ \dim_Q(X \times Y) &= \dim_Q X + \dim_Q Y, \\ \dim_{Q_p}(X \times Y) &= \max[\dim_{Z_p}(X \times Y) - 1, \dim_{Q_p} X + \dim_{Q_p} Y], \\ \dim_{R_p}(X \times Y) &= \min\{\max[\dim_Q(X \times Y), \dim_{Z_p}(X \times Y), \dim_{Q_p} X + \dim_{Q_p} Y + 1], \\ &\max[\dim_Q(X \times Y), \dim_{Z_p}(X \times Y), \dim_{R_p} X + \dim_{Q_p} Y, \dim_{Q_p} X + \dim_{R_p} Y]\}. \end{aligned}$$

$-d(X)$, and there do not exist compacta Y for which

$$\dim(X \times Y) < \dim X + \dim Y - d(X).$$

Theorem 3. $d(X) \leq d$ if and only if the following conditions are satisfied: 1) $\dim(X \times X_{Q_p}^{d+1}) \geq \dim X + \dim X_{Q_p}^{d+1} - d$ for every prime number p ; 2) $\dim(X \times X_Q^{d+1}) \geq \dim X + \dim X_Q^{d+1} - d$. In the case $d = 0$, it is sufficient that condition 1) alone be satisfied.

Proof. First of all, note that for $d = 0$ this theorem becomes the well-known theorem of V. G. Boltyanskii (⁴).

The necessity of the conditions is obvious.

Let us verify sufficiency. By Theorem 2, $\dim_{Q_p} X \geq \dim X - d$ and $\dim_Q X \geq \dim X - d$. Hence, and from inequalities (1), it follows that $\dim_G X \geq \dim X - d$ for any group $G \in \sigma$. By Bockstein's formulas, for any compactum Y we obtain $\dim(X \times Y) \geq \dim X + \dim Y - d$. Consequently, $d(X) \leq d$. The sufficiency of condition 1) for $d = 0$ follows from the fact that in this case, for any group $G \in \sigma$, we have $\dim_G X_{Q_p}^1 \leq \dim_G X_Q^1$. The theorem is proved.

We shall say that a compactum X is a compactum of ordinary type if $\dim X^n = n \dim X$ for every $n > 0$. If $\dim X^n = (n - 1) \dim X + 1$ for every $n > 0$, then we shall say that the compactum X is a compactum of special type.

Theorem 4. Every compactum X is either a compactum of ordinary type or a compactum of special type. A compactum X is a compactum of special type if and only if, for every prime number p ,

$$\dim(X \times X_{Z_p}^1) < \dim X + \dim X_{Z_p}^1,$$

$$\dim(X \times X_Q^1) < \dim X + \dim X_Q^1. \quad (2)$$

Proof. Suppose that inequalities (2) hold for the compactum X . By Theorem 2, $\dim_{Z_p} X \leq \dim X - 1$, $\dim_Q X \leq \dim X - 1$. For each p , by Bockstein's formulas, $\dim_{R_p} X^n \leq n(\dim X - 1) + 1$. For those p for which $\dim_{R_p} X = \dim X$, in view of inequalities (1) we have $\dim_{Q_p} X = \dim_{R_p} X - 1$. Therefore, for such p we have $\dim_{R_p} X^n = n(\dim X - 1) + 1$. Consequently, the compactum X is a compactum of special type.

Now let X be an arbitrary compactum for which inequalities (2) are not satisfied. By Theorem 2, either $\dim_Q X = \dim X$, or for some prime number p , $\dim_{Z_p} X = \dim X$. By Bockstein's formulas, $\dim X^n = \max_p \dim_{R_p} X^n = n \dim X$. The compactum X is a compactum of ordinary type. The theorem is proved.

The following assertion was first proved by Fary (⁷).

Corollary. For an arbitrary compactum X , the inequality $\dim(X \times X) \geq 2 \dim X - 1$ holds.

Note that the Boltyanskii compactum is a two-dimensional compactum of special type. The product of a compactum of special type with a dimensionally full-valued compactum is a compactum of special type.

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* A compactum X is dimensionally full-valued if and only if $d(X) = 0$. Always $d(X) \geq 0$.

Note: Figure translations are in progress. See original paper for figures.

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