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Abstract

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INVARIANTS OF LINEAR REPRESENTATIONS OF THE GROUP OF ROTATIONS OF THE PLANE AND THE CENTER PROBLEM

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In this note algebraic invariants ([1,2]) of the group of linear transformations of the space of coefficients of the system (1), induced by rotation of the phase plane, are studied. A simple method is given for obtaining a finite integral rational basis of these invariants (Theorem 1). With the aid of the constructed invariants, conditions are formulated for the existence of axes of symmetry, passing through the origin of coordinates, of the field of directions determined by the system under consideration in the phase plane (Theorems 2 and 3). In the cases when the problem arises of distinguishing a center from a focus ([3]), these conditions generalize known sufficient conditions for the existence of a center ([4–8]), under which the axes of symmetry are the coordinate axes or the bisectors of the coordinate angles. Theorem 4 makes it possible to judge the place of the conditions obtained in the system of all conditions for the existence of a center in those cases for which the center and focus problem has been solved completely ([9,10]).

1°. Consider the system of differential equations

$$\frac{dx}{dt} = \sum_{j+l\in A} c_{jl} x^j y^l, \qquad \frac{dy}{dt} = \sum_{j+l\in A} b_{jl} x^j y^l, \tag{1}$$

in which A is some finite set of distinct nonnegative integers, and c_{jl} and b_{jl} are real numbers.

We shall regard the aggregate of all coefficients c_{jl} and b_{jl} of the right-hand sides of the system (1) as a point s of a real vector space S. If in the plane XOY the coordinate axes are rotated through some angle φ ($0 \le \varphi < 2\pi$), then in the new coordinates x_1 and y_1 the system (1) takes the form

$$\frac{dx_1}{dt} = \sum_{j+l\in A} c_{jl}^{(1)} x_1^j y_1^l, \qquad \frac{dy}{dt} = \sum_{j+l\in A} b_{jl}^{(1)} x_1^j y_1^l. \tag{2}$$

We shall likewise regard the aggregate of coefficients of the right-hand sides of the system (2) as a point s_1 in S. All these coefficients are expressed linearly in terms of the coefficients of the right-hand sides of the original system (1). Thus, to a rotation of the coordinate system through the angle φ there corresponds a certain linear transformation L_φ of the space S into itself such that $s_1 = L_\varphi s$. The aggregate L of linear transformations L_φ for all $\varphi \in [0, 2\pi)$ forms a group, homomorphic to the group of rotations of the plane, i.e. is one of the possible linear representations of this group ([2,11]).

Consider the set $\{I(s)\}$, where $I(s)$ is any algebraic invariant (a.i.) of the group L over the field R of real numbers, i.e. a polynomial in the coordinates of the vector s over R such that $I(L_\varphi, s) \equiv I(s)$ for all $\varphi \in [0, 2\pi)$ and $s \in S$. One may also consider algebraic invariants of the group L over the field C of complex numbers. The real and imaginary parts of such an invariant will be algebraic invariants of L over R.

We set ourselves the task of indicating a simple method for constructing a finite integral rational basis for the set $\{I(s)\}$, i.e. such a finite set of algebraic invariants

$$I_1(s),\ I_2(s), \dots,\ I_\alpha(s), \tag{3}$$

that every invariant from $\{I(s)\}$ is an integral rational function of the invariants (3) (with coefficients from the same field over which the invariants $I(s)$ are considered). For this purpose we pass from the representation L to another linear representation of the group of rotations of the plane, whose objects are vectors of a complex vector space—

with coordinates $\eta_{jl} \equiv c_{jl} + ib_{jl}$ $(j + l \in A)$, and from it to an equivalent linear representation consisting of purely diagonal matrices.

2°. With the aid of the complex variable $w = x + iy$, we pass from system (1) to the equation

$$\frac{dw}{dt} = \sum_{j+l\in A} 2^{-j-l} z_{jl} \overline{w}^j w^l. \tag{4}$$

The coefficients z_{jl} of the right-hand side of (4) are determined by the equalities

$$z_{n-j,j} = \sum_{l=0}^{n} i^l R_{jl}^{(n)} \eta_{n-l,l}, \qquad \text{where} \quad R_{jl}^{(n)} = \sum_{\sigma} (-1)^\sigma \binom{l}{\sigma} \binom{n-l}{j-\sigma}, \tag{5}$$

where the summation here extends over all integers σ satisfying the conditions $0 \le \sigma \le l$ and $0 \le j - \sigma \le n - l$.

Let us introduce the complex space Z of vectors z with coordinates z_{jl} $(j+l \in A)$. As is seen from formulas (5), the real and imaginary parts of the coordinates of the vector z are expressed through the coordinates of the vector s linearly with integer coefficients. In this way a one-to-one correspondence $z = Ks$ ($s = K^{-1}z$) is established between the points of the spaces S and Z. To the group L of linear transformations L_φ of the space S there then corresponds a certain group U of linear transformations U_φ of the space Z, such that $U_\varphi z = KL_\varphi K^{-1}z$.

To obtain the matrix M_φ of the transformation U_φ, we make in (4) the substitution $w_1 = we^{-i\varphi}$, corresponding to a rotation through the angle φ of the coordinate axes in the plane XOY. We thereby arrive at the equation

$$\frac{dw_1}{dt} = \sum_{j+l\in A} 2^{-j-l} z_{jl}^{(1)} \overline{w}_1^j w_1^l,$$

in which the coordinates $z_{jl}^{(1)}$ of the vector Uz satisfy the relations:

$$z_{jl}^{(1)} = e^{(l-j-1)i\varphi} z_{jl} \qquad (j + l \in A). \tag{6}$$

Formulas (6) show that the matrix M_φ has the following diagonal form:

$$M_\varphi = [\dots, e^{-(n+1)i\varphi}, e^{-(n-1)i\varphi}, \dots, e^{(n-1)i\varphi}, \dots] \qquad (n \in A).$$

Take an arbitrary absolute invariant $I(s)$ (over C) of the group L, and consider the function $J(z) \equiv I(K^{-1}z)$, which is a certain polynomial in z_{jl} and $\overline{z}_{jl}$ (with complex coefficients). It is obvious that $J(z)$ is an invariant of the group U, i.e.,

$$J(U_\varphi, z) \equiv J(z) \qquad \text{for any } \varphi \in [0, 2\pi) \text{ and } z \in Z. \tag{7}$$

Conversely, if one takes some absolute invariant of the group U over C, i.e. a polynomial $J(z)$ over C in z_{jl} and $\overline{z}_{jl}$ satisfying condition (7), then the function $I(s) \equiv J(Ks)$ is an absolute invariant of the group L over C.

The problem now reduces to finding polynomials $j_r(z)$ ($r = 1, 2, \dots, \alpha$) forming a finite integral rational basis of the absolute invariants of the group U over C. Then the real and imaginary parts of the polynomials $I_r(s) \equiv J_r(Ks)$ ($r = 1, 2, \dots, \alpha$) will constitute such a basis of the absolute invariants of the group L over R.

Let E be an arbitrary set of distinct coordinates $z_1, z_2, \dots, z_m$ of the variable vector z, and let $e^{k_1 i\varphi}, e^{k_2 i\varphi}, \dots, e^{k_m i\varphi}$ be the corresponding proper numbers of the matrix M_φ. Denote by Z^E the space of vectors with coordinates $z_1, z_2, \dots, z_m$. Each Z^E can at the same time be considered as a subspace of the space Z invariant with respect to the group U. In connection with this it makes sense to speak of the transformations U_φ and their invariants on each of the subspaces Z^E. Denote by U_φ^E the linear transformation induced by the transformation U_φ on Z^E, by M_φ^E its matrix, and by U^E the group of transformations U_φ^E for $\varphi \in [0, 2\pi)$. Then

$$M_\varphi^E = [e^{k_1 i\varphi}, e^{k_2 i\varphi}, \dots, e^{k_m i\varphi}].$$

The integers $k_1, k_2, \dots, k_m$, independent of φ, will be called the **characteristic exponents** of the group of transformations U^E.

Theorem 1. Let $\alpha_r,\ \beta_r$ be arbitrary nonnegative integers such that: a) they form a nonzero solution of the equation

$$\sum_{r=1}^{m} (\alpha_r - \beta_r) k_r = 0$$

and b) there is no other nonzero solution of this equation in nonnegative integers $\alpha'_r,\ \beta'_r$ such that $\alpha'_r \leq \alpha_r,\ \beta'_r \leq \beta_r$ $(r = 1, 2, \dots, m)$.

Then the totality of all products of the form $\prod_{r=1}^{m} z_r^{\alpha_r} \bar{z}_r^{\beta_r}$ forms a finite integral rational basis of the algebraic invariants of the group U^E over R (and over C).

Using this theorem in the case when the set E contains all coordinates of the vector z $(Z^E = Z)$, it is easy to construct a finite integral rational basis

$$J_1(z),\ J_2(z), \dots, J_\alpha(z) \tag{8}$$

of algebraic invariants of the group U.

3°. Let us apply the above to the question of finding an **axis of symmetry** of the form $x \sin\varphi - y \cos\varphi = 0$ $(0 \leq \varphi \leq \pi)$ of the field of directions of the differential equation

$$\frac{dy}{dx} = -\sum_{j+l \in A} c_{jl} x^j y^l \Big/ \sum_{j+l \in A} b_{jl} x^j y^l. \tag{9}$$

System (1) determines in the phase plane XOY a family of orthogonal trajectories for the integral curves of equation (9). Therefore the axes of symmetry of the field of directions of equation (9) coincide with the axes of symmetry of the field of directions determined by system (1) in the phase plane XOY.

To each equation (9) there corresponds a point with coordinates b_{jl}, c_{jl} of the space S and a point $z = Ks$ of the space Z. Denote by $\widetilde{Z}$ the set of points of the space Z for which the numerator and denominator of the right-hand side of equation (9) have no common factors of nonzero degree and the condition

$$\sum_{j+l = \inf A} (b_{jl} y + c_{jl} x) x^j y^l \neq 0 \quad \text{for } x^2 + y^2 \neq 0 \tag{10}$$

is fulfilled.

Theorem 2. For the existence of an axis of symmetry of the field of directions of equation (9) passing through the origin, it is sufficient, and for $z \in \widetilde{Z}$ also necessary, that all algebraic invariants of the group U over R have real values at the point z.

The vanishing at the point z of the imaginary parts of all a.i. of the group U over R is, obviously, equivalent to the fulfillment of the conditions $\operatorname{Im} J_l(z) = 0$ $(l = 1, 2, \dots, \alpha)$. These conditions, however, are not independent, despite the fact that among the invariants of the basis (8) there are no "superfluous" ones (in the sense that none of them is an integral rational function of the others). Nevertheless there exists such a minimal number $\alpha'(A) \leq \alpha/2$ that the vanishing of the imaginary parts of all invariants (8) reduces to the equivalent system of independent conditions

$$\operatorname{Im} J_{r_1}(z) = \operatorname{Im} J_{r_2}(z) = \dots = \operatorname{Im} J_{r_{\alpha'}}(z) = 0. \tag{11}$$

Fix a point $z \in Z$ and denote by E_z the set of those coordinates of vectors from Z which at this point are not equal to zero.

Theorem 3. If $z \in \widetilde{Z}$ and conditions (11) are fulfilled, then the number of distinct axes of symmetry of the field of directions of equation (9) passing through the origin is equal to the greatest common divisor of the absolute values of the characteristic exponents of the group U^{E_z} (and, in particular, is equal to ∞ if all characteristic exponents are equal to zero).

4°. As is known ([3]), when condition (10) is fulfilled, which guarantees the absence of exceptional directions, the origin is for

equation (9) a singular point of center or focus type. In this case conditions (11) will be sufficient conditions for the existence of a center. They generalize the known conditions ([4–8]), under which the coordinate axes or the bisectors of the coordinate angles are axes of symmetry of the field of directions of equation (9).

The generally known sufficient conditions for a center for equation (9) under condition (10) are given by the system of equalities

$$l z_{j-1,l} = \bar{j} z_{l-1,j} \qquad (j + l - 1 \in A), \tag{12}$$

under whose fulfillment (9) is an equation in total differentials. Denote by Z' the set of points of the space Z for which at least one of conditions (11) is not fulfilled. In questions connected with finding conditions for the existence of a center, it is sufficient to restrict oneself to studying system (11) for $z \in Z'$.

Conditions (11) for $z \in Z'$ can be fulfilled only when at least one of the differences $z_{j-1,l} \equiv l z_{j-1,l} - j \bar{z}_{l-1,j}$, with $l > j$, is different from zero. Denote by z' the vector with coordinates z_{jl} for $l \leq j+1$ and z'_{il} for $l > j+1$. It turns out that system (11) is always equivalent to the system $\operatorname{Im} J_{r_l}(z') = 0$ $(l = 1, 2, \dots, \alpha')$.

The latter conditions, while independent on the whole space Z, may prove dependent on the subspace Z'. There exists, however, such a minimal number $\alpha''(A) \leq \alpha'$ that system (11) is equivalent on the subspace Z' to the system

$$\operatorname{Im} J_{\rho_1}(z') = \operatorname{Im} J_{\rho_2}(z') = \cdots = \operatorname{Im} J_{\rho_{\alpha''}}(z') = 0. \tag{13}$$

These conditions, together with (10), are sufficient for the presence of a center at the origin for equation (9). They are always fulfilled when conditions (11) are fulfilled, but, generally speaking, are simpler than the latter.

Let us note that for the equation

$$\frac{dy}{dx} = -\left(x + \sum_{j+l\in A'} c_{jl}x^j y^l\right) \Big/ \left(y + \sum_{j+l\in A'} b_{jl}x^j y^l\right), \tag{14}$$

in which A' is some finite set of distinct natural numbers not containing one, conditions (10) are fulfilled, while conditions (11), (12), and (13) are equivalent to the corresponding conditions for equation (9) with $A = A'$. A simple calculation shows, for example, that for $A' = \{2\}$ conditions (13) can be written in the form $\operatorname{Im}(z_{11}z'_{02}) = \operatorname{Im}(z_{20}z'^3_{02}) = 0$, and for $A' = \{3\}$ in the form $\operatorname{Im} z_{12} = \operatorname{Im}(z_{21}z'_{03}) = \operatorname{Im}(z_{30}z'^2_{03}) = 0$. The place of these conditions in the system of all conditions for the existence of a center is seen from the following theorem, which is obtained by expressing the center conditions given in ([9,10]) in terms of the coordinates of the vector z'.

Theorem 4. *For the equation (14) to have a center at the origin when $A' = \{2\}$, it is necessary and sufficient that at least one of the following three series of conditions be fulfilled:*

$$1)\ z_{11} = 0; \qquad 2)\ \operatorname{Im}(z_{11}z'_{02}) = \operatorname{Im}(z_{20}z'^3_{02}) = 0;$$
$$3)\ 5\bar{z}_{11} + z_{02} = 5|z_{20}| - |z'_{02}| = 0,$$

and when $A' = \{3\}$, at least one of the following two series of conditions:

$$1)\ \operatorname{Im} z_{12} = \operatorname{Im}(z_{21}z'_{03}) = \operatorname{Im}(z_{30}z'^2_{03}) = 0;$$
$$2)\ z_{12} = 10z_{21} + \bar{z}'_{03} = z^2_{30}\bar{z}_{21} + 4z^3_{21} = 0.$$

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Note: Figure translations are in progress. See original paper for figures.