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Ya. G. BERKOVICH

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Abstract

Full Text

MATHEMATICS

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A CHARACTERIZATION OF CERTAIN CLASSES OF FINITE GROUPS

(Presented by Academician A. I. Mal' tsev, 25 II 1963)

§ 1. In recent years numerous works have appeared devoted to the characterization of finite groups (see ([1–8])). The present note is devoted to an analogous question.

Only finite groups are considered. The following notation and concepts are used. (G) is the order of the group G. π is a certain set of primes. $\pi(G)$ is the set of all prime divisors of (G). π_G is the closure of π in G, i.e. the set containing all prime divisors of each πd-subgroup of the group G. The set π is closed in G if it coincides with its minimal closure in G (we note that π_G is not uniquely defined). $(G)_\pi$ is the greatest π-divisor of (G). If

$$(G)_\pi = \prod_1^k p_i^{\alpha_i}$$

is the canonical decomposition of $(G)_\pi > 1$ into prime factors, then

$$\lambda_\pi(G) = \sum_1^k \alpha_i, \qquad \tau_\pi(G) = k,$$

if $(G)_\pi > 1$; if $(G)_\pi = 1$, then $\lambda_\pi(G) = \tau_\pi(G) = 0$.

Two groups are called *isoordinal* if their orders are equal. The totality of subgroups of a group G that are isoordinal with a given subgroup of it is called a *class of isoordinal subgroups*. A class of isoordinal subgroups is called *noninvariant* if it contains at least one noninvariant subgroup ([14]). If a class of isoordinal subgroups contains at least one (P)-subgroup (as groups possessing the group-theoretic property (P) are called), then it is called a *class of isoordinal (P)-subgroups*.

We shall agree, in the case $\pi \supseteq \pi(G)$, to omit the symbol π in all notation. $\overline{S}_k(G)$ is the totality of all those noninvariant soluble subgroups H of G for which $\tau(H) \le k$, k being a fixed nonnegative integer. $sn_\pi(G)$ is the number of such classes of isoordinal πd-subgroups of the group G, each of which contains at least

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one subgroup from $\overline{S}_3(G)$. $r_k(G)$ is the number of such classes of isomorphic subgroups of the group G, each of which contains at least one nonnilpotent subgroup from $\overline{S}_k(G)$. $N_G(H)$ is the normalizer of the subgroup H in the group G.

A group of type S is a nonnilpotent group all of whose proper subgroups are nilpotent ([15,16]). A group G is called *primary* if $\tau(G) = 1$. A group of type A is a nonnilpotent group all of whose proper subgroups are primary ([16]). H_G is the intersection of all subgroups conjugate to H in G ([13]). The definition of the groups $LF(2,p)$ and $SL(2,p)$, where p is a prime number, can be found in ([2]).

Subgroups H and F form a nilpotent pair if they satisfy at least one of the following conditions:

a) $H \subseteq F$; b) $F \subseteq H$; c) $N_H(H \cap F) \neq H \cap F \neq N_F(H \cap F)$.

A group G is called *quasinilpotent* if any two of its subgroups form a nilpotent pair. It is obvious that groups,

all of whose proper subgroups are nilpotent are quasinilpotent.

§ 2. **Theorem 1.** *Let G be a nonnilpotent quasinilpotent group. Then it has the following properties:*

1. $\tau(G) = 2$. *Let $|G| = p^\alpha q^\beta$, and suppose that the subgroup of order p^α is noninvariant in G.*
2. *The commutant of the group G has order q^β.*
3. *The p-Sylow subgroup of the group G is cyclic; the subgroups of order $p^{\alpha-1}$ belong to the center of the group G.*
4. *All normal divisors of the group G are nilpotent. All subgroups of the group G whose order is divisible by p^α are noninvariant.*
5. *If H is a noninvariant maximal subgroup of the group G, then G/H_G is of type A. All noninvariant maximal subgroups of the group G are isomorphic and have order $p^\alpha q^{\beta-b}$, where b is the least natural number satisfying the congruence $q^b \equiv 1 \pmod{p}$.*
6. *If H is a noninvariant maximal subgroup of the group G, then $H = \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of the group G. In particular, $G/\Phi(G)$ is of type A.*
7. *G is of type S if and only if $\beta < 2b$.*

It is unknown whether nonnilpotent quasinilpotent groups will always be groups of type S.

Theorem 2. *If all nonnilpotent subgroups from $\overline{S}_3(G)$ are of type S, then only one of the following possibilities occurs:*

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1. *G is a solvable group.*
2. $G \cong LF(2,5)$.
3. $G \cong S, L(2,5)$.

Theorem 2 strengthens a result of M. Suzuki ([3]), p. 695, as well as a theorem of Z. Janko ([11]).

Theorem 3. *Let G be a nonsolvable πd-group, π a nonempty set of prime numbers. If*

$$sn_\pi(G) < \lambda_\pi(G) + \tau_\pi(G) + 1,$$

then G is a simple group.

Theorem 4. *Let G be a nonsolvable group and*

$$sn(G) < \lambda(G) + \tau(G) + 1.$$

Then either $G \cong LF(2,5)$, *or* $G \cong LF(2,11)$.

Theorem 4 generalizes a result of N. Ito ([9]).

If we impose no restrictions on the orders of the subgroups under consideration, then instead of the symbols $sn_\pi(G)$, $\overline{S}_k(G)$, $r_k(G)$ we shall write respectively $s_\pi(G)$, $\overline{S}(G)$, $r(G)$. A nonabelian group all of whose maximal subgroups are abelian will be called a group of type M.

Theorem 5. *Let G be a nonsolvable group of even order. If all nonabelian* $2d$*-subgroups from* $\overline{S}(G)$ *are of type M, then G is isomorphic to the icosahedral group.*

Theorem 6. *Let G be a nonsolvable group, p the least prime divisor of* $|G|$, $\pi_G = \pi \ni p$. *If*

$$sn_\pi(G) < 2\tau_\pi(G) + 2,$$

then either $G \cong LF(2,5)$, *or* $G \cong LF(2,11)$.

Theorem 5 generalizes a result of L. Redei ([6]), and Theorem 6—a result of N. Ito ([9]).

Theorem 7. *If G is a nonsolvable group and* $r(G) < \tau(G) + 1$, *then either* $G \cong LF(2,5)$, *or* $G \cong SL(2,5)$.

Theorem 7 generalizes a theorem of D. P. Kolyankovsky ([17]). It is of interest to determine whether an analogue of Theorem 7 holds for a theorem of V. I. Sergienko ([18]).

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Theorem 8. Let $H \neq 1$ be a solvable normal divisor of a nonsolvable group G. If $sn(G) < 3\tau(G) + 2$, then only one of the following possibilities occurs:

1. $G \cong SL(2, 5)$.
2. $G \cong SL(2, 11)$.
3. $G = G_1 \times H$, where $G_1 \cong LF(2, 5)$, $(H) = 2$.

From Theorem 8 follows the result of Ito-Szep ([10]).

Let $H \neq 1$ be a normal divisor of the group G, and let

$$G \supset \cdots \supset H \supset H_1 \supset \cdots \supset H_k = 1$$

be a principal series of the group G passing through H. Then put $k = l_G(H)$.

Theorem 9. If $sn(G) = 2\tau(G) + 2$, then G is a solvable group.

Theorem 10. Let $H \neq 1$ be a solvable normal divisor of a nonsolvable group G. If

$$s(G) < 2\tau(G) + 3 + l_G(H)[\tau(G) - 1],$$

then the group G has the following properties:

1. H is a 2-subgroup coinciding with the center of the group G. If $(H) > 4$, then $H = \Phi(G)$.
2. Either $G/H \cong LF(2, 5)$, or $G/H \cong LF(2, 11)$. If $G/H \cong LF(2, 11)$, then $G \cong SL(2, 11)$.

We shall call a subgroup H of a group G **completely noninvariant** if all subgroups of G that are isomorphic to H are noninvariant in G.

Theorem 11. In Theorems 3, 4, 6, 8, 9, noninvariance may be replaced by complete noninvariance.

Theorem 12. Let P, Q, R be pairwise nonisomorphic maximal subgroups of the group G. If P, Q, R are nilpotent, then G is nilpotent.

Theorem 13. Let all nonnilpotent maximal subgroups of the group G be isomorphic. Then G is solvable and $\tau(G) \leq 3$. If $\tau(G) = 3$, then $G = P \times H$, where H is of type S, and P is a Sylow subgroup of the group G.

Theorem 14. If G is a nonsolvable group, then it contains at least three noninvariant, pairwise nonisomorphic second maximal subgroups.

Let H be an arbitrary subgroup of the group G with $\lambda(H) = 2$. If all maximal chains from G to H have one and the same length, then we shall call G a J_2-group.

Theorem 15. Let G be a J_2-group. Then G is either supersolvable, or of type A.

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Theorem 15 generalizes a known theorem of K. Iwasawa on the supersolvability of J-groups ([12]).

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