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Abstract

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ON A BOUNDARY-VALUE PROBLEM

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Consider, in the domain $Q_T = (0 < t \leq T,\ 0 < x < 1)$, the differential equation

$$Pu + P_1(t,x)u = f(t,x), \tag{1}$$

where $Pu \equiv u_t + (-1)^m u_{x^{2m+1}}$; $P_1(t,x)u \equiv a_0(t,x)u_{x^{2m}} + \dots + a_{2m}(t,x)u$; $a_0(t,x) \in C^{1,2m}_{t,x}(\overline{Q}_T)$, $a_k(t,x)$, $k = 1, \dots, 2m$, are summable, bounded in $\overline{Q}_T$ functions; $f(t,x) \in L^2(Q_T)$.

We shall be interested in the question of unique solvability of equation (1) under the conditions

$$u(0,x) = 0; \tag{2}$$

$$u_{x^k}(t,0) = u_{x^l}(t,1) = 0, \qquad k = 0, \dots, m; \quad l = 0, \dots, m-1. \tag{3}$$

(If condition (3) is satisfied for $u(t,x)$, we shall say that $u \in \Gamma$.)

A similar problem in the case $m = 1$ was considered by Cattabriga (^{1}), who obtained existence and uniqueness theorems for this problem in some special cases.

Theorem 1. Let

$$u(t,x) = \sum_{k=1}^{N} a_k(t) X_k(x), \tag{4}$$

where $a_k(t) \in C^1(0,T)$, $a_k(0) = 0$, $k = 1, \dots, N$; $\{X_k\}_1^\infty$ is a complete (^{2}) system of eigenfunctions and associated functions of the operator

$$(-1)^{m+1} \frac{d^{2m+1}}{dx^{2m+1}} X(x), \qquad x \in (0,1), \quad X \in \Gamma.$$

Then there exist positive constants K and σ_0 such that, for all $\sigma > \sigma_0$, any $T > 0$, and any function $u(t, x)$ of the form (4), the inequalities

$$\|u_{x^l}\|_{L^2(Q_T)} \leq K\sigma^{-\frac{2m-l}{2m+1}} e^{\sigma T} \|Pu\|_{L^2(Q_T)}, \qquad l = 0, \dots, 2m. \tag{5}$$

hold.

The constants K and σ_0 depend neither on T nor on N.

We extend the given function $u(t, x)$ to the domain $(T \leq t < \infty,\ 0 \leq x \leq 1)$ by a function $z(t, x)$, which we construct as follows:

$$z(t, x) \in C^{1,2m+1}_{t,x} \quad \text{in the domain } (T \leq t < \infty,\ 0 \leq x \leq 1),$$

$$Pz = 0, \qquad z \in \Gamma, \qquad z(T, x) = u(T, x), \qquad |z(t, x)| \leq Ce^{\sigma_0 t}$$

with some constants C and $\sigma_0 > 0$.

Then the function $u(t, x) \in C^{1,2m+1}_{t,x}$ in the domains $(0 < t \leq T,\ 0 \leq x \leq 1)$ and $(T \leq t < \infty,\ 0 \leq x \leq 1)$ is continuous at $t = T$, and $|u(t, x)| \leq C_1 e^{\sigma_0 t}$ with some constant C_1.

Applying now the Laplace transform with respect to t to the identity

$$u_t + (-1)^m u_{x^{2m+1}} = f \equiv Pu,$$

we obtain that the function

$$v(\lambda, x) = \int_0^\infty u(t, x) e^{-\lambda t}\, dt$$

must satisfy the equation

$$\lambda v + (-1)^m v_{x^{2m+1}} = F(\lambda, x), \qquad v \in \Gamma, \qquad F(\lambda, x) = \int_0^\infty Pu\, e^{-\lambda t}\, dt,$$

$$\lambda = \sigma + i\gamma, \qquad \sigma > \sigma_0 > 0.$$

Consider the functions $w_1(\lambda, x) = v(\lambda, x)\varphi(x)$ and $w_2(\lambda, x) = v(\lambda, x)\varphi(1 - x)$, where $\varphi(x)$ is equal to 1 for $x \in [0, \frac{1}{2}]$, is equal to 0 for $x \in [\frac{3}{4}, 1]$, $\varphi(x) \in C^\infty(0, 1)$, $|\varphi(x)| \leqslant 1$. We note that $v(\lambda, x) = w_1(\lambda, x)$ for $0 \leqslant x \leqslant \frac{1}{2}$ and $v(\lambda, x) = w_2(\lambda, x)$ for $\frac{1}{2} \leqslant x \leqslant 1$. Taking $w_1(\lambda, x)$ to be equal to zero for $1 < x < \infty$, we may regard it on the half-axis $(0, \infty)$ as a solution of the equation

$$\lambda w_1 + (-1)^m \frac{d^{2m+1}}{dx^{2m+1}} w_1 = h_1(\lambda, x), \tag{6}$$

where

$$h_1(\lambda, x) = F(\lambda, x)\varphi(x) + (-1)^m \sum_{k=0}^{2m} C_{2m+1}^k v_{x^k}(\lambda, x)\varphi_{x^{2m+1-k}}(x),$$

and $w_1(\lambda, x)$ satisfies the conditions

$$w_{1x^k}(\lambda, 0) = w_{1x^k}(\lambda, \infty) = 0, \qquad k = 0, \ldots, m; \quad l = 0, \ldots, m-1.$$

Similarly, the function $w_2(\lambda, x)$, extended by zero for $x < 0$, is regarded on the half-axis $(-\infty, 1)$ as a solution of the equation

$$\lambda w_2 + (-1)^m \frac{d^{2m+1}}{dx^{2m-1}} w_2 = h_2(\lambda, x) \equiv F(\lambda, x)\varphi(1-x) +$$

$$+ \sum_{k=0}^{2m} (-1)^{m+1-k} C_{2m+1}^k v_{x^k}(\lambda, x)\varphi_{x^{2m+1-k}}(1-x) \tag{7}$$

under the conditions

$$w_{2x^k}(\lambda, -\infty) = w_{2x^l}(\lambda, 1) = 0, \qquad k = 0, \ldots, m; \quad l = 0, \ldots, m-1.$$

With the aid of representations of the solutions of problems (6) and (7) in terms of the corresponding Green' s functions, one can prove the validity of the following estimates: for m odd,

$$\|v_{x^k}\|_{(0,1/2)} = \|w_{1x^k}\|_{(0,1/2)} \leqslant C_2 \sigma^{-\frac{2m-k}{2m+1}} \left(\|F\|_{(0,1)} + C_3 \sigma^{-\frac{1}{2(2m+1)}} \sum_{l=0}^{2m} \|v_{x^l}\|_{(1/2,1)} \right), \tag{8}$$

$$\|v_{x^k}\|_{(1/2,1)} = \|w_{2x^k}\|_{(1/2,1)} \leqslant C_2 \sigma^{-\frac{2m-k}{2m+1}} \left(\|F\|_{(0,1)} + C_3 \sum_{l=0}^{2m} \|v_{x^l}\|_{(0,1/2)} \right); \tag{9}$$

for m even,

$$\|v_{x^k}\|_{(0,1/2)} = \|w_{1x^k}\|_{(0,1/2)} \leqslant C_2\sigma^{-\frac{2m-k}{2m+1}} \left(\|F\|_{(0,1)} + C_3 \sum_{l=0}^{2m} \|v_{x^l}\|_{(1/2,1)} \right), \quad (10)$$

$$\|v_{x^k}\|_{(1/2,1)} = \|w_{2x^k}\|_{(1/2,1)} \leqslant C_2\sigma^{-\frac{2m-k}{2m+1}} \left(\|F\|_{(0,1)} + C_3\sigma^{-\frac{1}{2(2m+1)}} \sum_{l=0}^{2m} \|v_{x^l}\|_{(0,1/2)} \right), \quad (11)$$

$$k = 0, 1, \dots, 2m.$$

Here $\|y\|_{(a,b)}$ denotes, as usual,

$$\left(\int_a^b |y(x)|^2 \, dx \right)^{1/2}.$$

Take σ sufficiently large and solve (8)–(9) (or (10)–(11)) with respect to $\|v_{x^k}\|_{(0,1/2)}$, $\|v_{x^k}\|_{(1/2,1)}$, $k = 0, \dots, 2m$. Then, for even as well,

and for odd m

$$\|v_{x^k}\|_{(0,1)} \leq K\sigma^{-\frac{2m-k}{2m+1}} \|F\|_{(0,1)}, \qquad k = 0, \dots, 2m. \quad (12)$$

The estimates (5) are now easily obtained with the aid of Parseval' s equality from the inequalities (12).

It follows from Theorem 1 that on the set of functions (4) one can introduce the scalar product

$$[u, v] = (Pu, Pv)_{\mathcal{L}^2(Q_T)}. \quad (13)$$

The Hilbert space obtained by completing this set in the metric (13) will be denoted by $H(Q_T)$. The inequalities (5) show that the embedding $H(Q_T) \subset W^{0,2m}_{t,x}(Q_T)$ is valid.

Definition. A function $u(t,x) \in H(Q_T)$ is called a **generalized solution** of problem (1)–(3) if there exists a sequence of functions $u_n(t,x)$ of the form (4), converging as $n \to \infty$ to $u(t,x)$ in the norm of $H(Q_T)$, and such that

$$\lim_{n\to\infty} \|Pu_n + P_1(t,x)u_n - f(t,x)\|_{\mathcal{L}^2(Q_T)} = 0.$$

It is easy to see that the generalized solution thus defined satisfies the integral identity

$$-(u, w_t)_{\mathcal{L}^2(Q_T)} + (-1)^{m-1}(u_{x^{2m}}, w_x)_{\mathcal{L}^2(Q_T)} + (P_1 u, w)_{\mathcal{L}^2(Q_T)} = (f, w)_{\mathcal{L}^2(Q_T)}$$

for any function $w \in \dot{W}_2^1(Q_T)$, $w(T, x) = w(t, 0) = w(t, 1) = 0$.

Theorem 2. *For every function $f(t, x) \in \mathcal{L}^2(Q_T)$ there exists a unique generalized solution of problem (1)–(3).*

Lemma. *For every function $f(t, x) \in \mathcal{L}^2(Q_T)$ there exists a unique generalized solution of the problem*

$$Pu = f, \qquad u(0, x) = 0, \qquad u \in \Gamma.$$

The validity of the lemma follows from Theorem 1 and the completeness of the system of functions $\{X_k\}_1^\infty$ in $\mathcal{L}^2(0 < x < 1)$ (^{2}).

Remark. After the substitution

$$u(t, x) = v(t, x) \exp\left[(-1)^{m+1} \int_0^x a(t, \xi)\, d\xi\right]$$

for the function $v(t, x)$ one obtains an equation of the same form as (1), but with $a_0(t, x) = 0$. Therefore it suffices to carry out the proof of Theorem 2 for the equation

$$Pu + P_2(t, x)u = f, \tag{14}$$

$$P_2(t, x)u = a_1(t, x)u_{x^{2m-1}} + \dots + a_{2m}(t, x)u; \qquad a_k(t, x),\ k = 1, \dots, 2m,$$

summable bounded functions in $\overline{Q}_T$.

From the lemma follows the existence of the operator $P^{-1} : \mathcal{L}^2(Q_T) \to H(Q_T)$, and hence also the existence of the operator $A = P^{-1}P_2 : H(Q_T) \to H(Q_T)$. In this case problem (14), (2), (3) turns out to be equivalent to the operator equation in the space $H(Q_T)$

$$u + Au = h = P^{-1}f. \tag{15}$$

Using the estimates (5), one can show that A is a completely continuous operator, i.e. equation (15) is of Fredholm type. Therefore, to complete the proof of Theorem 2 it is enough to show the uniqueness of the trivial solution of the homogeneous equation

$$Pu + P_2 u = 0, \qquad u(0,x) = 0, \qquad u \in \Gamma. \tag{16}$$

Let the function $u(t,x)$ be a solution of this problem in the domain Q_T; then $u(t,x)$ is also a solution in any smaller domain Q_{T_1}, $0 < T_1 \leq T$.

From Theorem 1 we have

$$\|u\|_{H(Q_{T_1})} = \|Pu\|_{\mathcal{L}^2(Q_{T_1})} = \|P_2 u\|_{\mathcal{L}^2(Q_{T_1})} \leq M \sum_{k=0}^{2m-1} \|u_{x^k}\|_{\mathcal{L}^2(Q_{T_1})} \leq$$

$$\leq MK\sigma^{-\frac{1}{2m+1}} \sum_{k=0}^{2m-1} \sigma^{-\frac{2m-k-1}{2m+1}} e^{\sigma T_1} \|u\|_{H(Q_{T_1})}, \tag{17}$$

where M is the maximum of the moduli of the coefficients of the operator P_2.

Choose σ so large that

$$MK\sigma^{-\frac{1}{2m+1}} \sum_{k=0}^{2m-1} \sigma^{-\frac{2m-k-1}{2m+1}} < \frac{1}{4}.$$

Then, for $T_1 < \frac{1}{\sigma} \ln 3$, inequality (17) is possible only when $\|u\|_{H(Q_{T_1})} = 0$. That is, $u = 0$ in the domain $(0 < t < T,\ 0 < x < 1)$.

By the definition of a generalized solution, there exists a sequence of smooth functions $u_n(t,x)$,

$$u_n(t,x) = \sum_{k=1}^{N_n} a_k(t) X_k(x), \qquad u_n(0,x) = 0,$$

converging in $H(Q_T)$ to the function $u(t,x)$ and such that

$$\lim_{n\to\infty} \|Pu_n + P_2(t,x) u_n\|_{\mathcal{L}^2(Q_T)} = 0.$$

It can be shown that the functions $v_n(t,x) = u_n(t,x)\psi(t)$, where $\psi(t) \in C^1(0,T)$, $\psi(t) = 0$ for $0 \leq t \leq T_1/2$; $\psi(t) = 1$ for $T_1/2 \leq t \leq T_1$, $|\psi(t)| \leq 1$, form a sequence of the same type as $\{u_n\}$. Moreover, $v_n(T_1/2, x) = 0$.

Thus, the sequence of functions $v_n^{(1)}(t,x) = v_n(t + T_1/2, x)$ defines in the space $H(Q_{T-T_1/2})$ the function

$$u^{(1)}(t,x) = \lim_{n\to\infty} v_n^{(1)}(t,x) \quad (\text{in } H(Q_{T-T_1/2})),$$

which is, in the domain $Q_{T-T_1/2}$, a generalized solution of the problem

$$Pu^{(1)} + P_2\left(t + \frac{T_1}{2}, x\right) u^{(1)} = 0, \qquad u^{(1)}(0,x) = 0, \qquad u^{(1)} \in \Gamma.$$

Applying inequality (17) again, we obtain

$$v^{(1)}(t,x) = 0 \qquad \text{for } 0 < t < T_1,$$

i.e.

$$u(t,x) = 0 \qquad \text{for } 0 < t < \frac{3}{2}T_1.$$

Taking, instead of $\psi(t)$, a function $\psi_1(t)$ equal to 0 for $0 \le t \le T_1$ and equal to 1 for $\frac{3}{2}T_1 \le t \le T$, $|\psi_1(t)| \le 1$, $\psi_1(t) \in C^1(0,T)$, and repeating all the preceding arguments, we obtain $u(t,x) = 0$ in the domain $(0 < t < 2T_1,\ 0 < x < 1)$. It is clear that in this way, after a finite number of steps, we prove that $u = 0$ in the entire domain Q_T, as was required.

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Note: Figure translations are in progress. See original paper for figures.

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