ON THE NATURE OF THE CONTACT OF THE FREE SURFACE OF A LIQUID WITH A SOLID BOUNDARY IN THE PROBLEM OF WEDGE PENETRATION

1963

SovietRxiv

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

MECHANICS

Z. N. DOBROVOL' SKAYA

ON THE NATURE OF THE CONTACT OF THE FREE SURFACE OF A LIQUID WITH A SOLID BOUNDARY IN THE PROBLEM OF WEDGE PENETRATION

(Presented by Academician A. A. Dorodnitsyn, 14 VI 1963)

The problem of the vertical penetration, with constant velocity v_0, of a wedge of arbitrary aperture angle 2α into a half-space occupied by an ideal incompressible and weightless liquid is considered in a nonlinear formulation.

The problem of wedge (cone) penetration, as well as the analogous problem of the impact of a liquid wedge on a solid boundary, has been considered in papers ([1–8]) and others. The study of these problems in a nonlinear formulation was reduced mainly to prescribing in advance the shape of the unknown free surface of the liquid, or to prescribing the missing conditions for functions introduced in solving the boundary-value problem ([4]); the question of the nature of the contact of the free surface of the liquid with the solid boundary was not investigated in any of the papers.

Fig. 1

In the present note the problem under investigation is reduced to finding a function $\zeta(w)$, analytic in the upper half-plane $\operatorname{Im} w > 0$, which maps the upper half-plane $\operatorname{Im} w > 0$ onto the region occupied by the liquid in the plane $\zeta = \xi + i\eta$ ($\xi = x/v_0 t,\ \eta = y/v_0 t$). The investigation of the functional equation for the function $\zeta(w)$ in a neighborhood of the point $w = 0$ (in the ζ-plane this is the point at which the free surface of the liquid meets the wedge) makes it possible to establish its singularity at this point and thereby to clarify the nature of the contact of the free surface with the solid boundary.

1. The motion of the liquid arising as a result of the penetration of the wedge is potential. The velocity potential $\varphi(x, y, t)$ is a function harmonic in the variables x, y in the region $ABCD$ (Fig. 1). On the free surface of the liquid, whose shape is unknown and must be found in the course of solving

the problem, the condition of constant pressure must be satisfied, as well as the kinematic condition

$$\frac{\partial\varphi}{\partial y} - \frac{\partial f_0}{\partial x}\frac{\partial\varphi}{\partial x} - \frac{\partial f_0}{\partial t} = 0, \tag{1,1}$$

where $y = f_0(x,t)$ is the equation of the unknown free boundary of the liquid. The function $\varphi(x,y,t)$ must also satisfy the following boundary conditions:

$$\frac{\partial\varphi}{\partial n} = v_0 \sin\alpha \quad \text{on the solid boundary } AB, \tag{1,2}$$

$$\frac{\partial\varphi}{\partial x} = 0 \quad \text{on the line of symmetry } AD; \tag{1,3}$$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = 0 \quad \text{as } |z| \to \infty$$

and the initial condition

$$\varphi(x,y,0) = 0.$$

2. Introduce the self-similar variables $\xi = x/v_0 t,\ \eta = y/v_0 t$. Then

$$\varphi(x,y,t) = v_0^2 t\, \Phi(\xi,\eta),$$

where $\Phi(\xi,\eta)$ is a harmonic function of the variables ξ,η. The equation of the free boundary will have the form $\eta = \eta(\xi)$. Conditions (1,1), (1,2), (1,3) pass respectively into the conditions

$$\frac{\partial\Phi}{\partial\eta} - \eta'(\xi)\frac{\partial\Phi}{\partial\xi} + \xi\eta'(\xi) - \eta(\xi) = 0 \quad \text{on } BC, \tag{2,1}$$

$$\frac{\partial\Phi}{\partial\xi}\cos\alpha - \frac{\partial\Phi}{\partial\eta}\sin\alpha = \sin\alpha \quad \text{on } AB, \tag{2,2}$$

$$\frac{\partial\Phi}{\partial\xi} = 0 \quad \text{on } AD. \tag{2,3}$$

In equation (2,1), ξ,η are the coordinates of points of the unknown curve $\eta = \eta(\xi)$.

3. Let $V(\zeta)$ be the complex velocity potential in the plane $\zeta = \xi + i\eta$. Introduce the plane of the auxiliary variable $w = u + iv$ and consider an analytic function $\zeta = \zeta(w)$ in the upper half-plane $\operatorname{Im} w > 0$, mapping the upper half-plane $\operatorname{Im} w > 0$ onto the region occupied by the fluid in the ζ-plane (Fig. 2), in such a way that the unknown free boundary of the fluid corresponds to the negative real semiaxis $u < 0$ in the w-plane (Fig. 3). The boundary conditions (2,1)–(2,3), written on the corresponding intervals of the real axis u of the w-plane, after certain transformations, can be reduced to the form

[Fig. 2 and Fig. 3]

$$\operatorname{Re}\,[\,iV'(w)\,] = \operatorname{Re}\,[\,i\zeta'(u)\overline{\zeta(u)}\,], \qquad -\infty < u \le 0, \tag{3,1}$$

$$\operatorname{Re}\,[\,iV'(w)\,] = \sin\alpha\,|\zeta'(u)|, \qquad 0 \le u \le 1, \tag{3,2}$$

$$\operatorname{Re}\,[\,iV'(w)\,] = 0, \qquad 1 \le u < +\infty. \tag{3,3}$$

In obtaining conditions (3,2), (3,3), the fact was used that the argument of the function $\zeta'(w)$ for $0 \le u \le 1$ and $1 \le u < +\infty$ is known. Conditions (3,1)–(3,3) make it possible to express, with the aid of the Schwarz integral for the upper half-plane, the complex potential $V(w)$ through the mapping function $\zeta(w)$:

$$V'(w) = -\frac{1}{\pi}\left[\int_{-\infty}^{0} \operatorname{Re}(i\zeta'\overline{\zeta})\,\frac{du}{u-w} + \sin\alpha \int_0^1 |\zeta'|\,\frac{du}{u-w}\right]. \tag{3,4}$$

Thus, the problem is reduced to finding the mapping function $\zeta(w)$.

4. To construct the equation satisfied by $\zeta(w)$, introduce the Wagner function ([1])

$$h = \int_{\infty}^{\zeta} \sqrt{\frac{d^2V(\zeta)}{d\zeta^2}}\,d\zeta. \tag{4,1}$$

The argument of the function $h(\zeta)$ is known everywhere on the boundary of the region $ABCD$ of the ζ-plane:

$$\arg[h(\zeta)] = \begin{cases} \pm\dfrac{\pi}{4}, & \text{on } CB, \\ 0, & \text{on } BA, \\ \pm\dfrac{\pi}{2}, & \text{on } AD. \end{cases}$$

Therefore, in the plane of values of this function, the domain occupied by the fluid is known. It is a closed triangle (Fig. 4).

Let us note that the value of the argument of the function $h(\zeta)$ on the free surface BC has been obtained as a consequence of the condition of constant pressure.

The function

$$h = ic_0 \int_{-\infty}^{w} w^{-3/4}(w-1)^{-1/2}\,dw \qquad (c_0 \text{ is a real constant}) \tag{4,2}$$

maps the upper half-plane $\operatorname{Im} w > 0$ (Fig. 3) onto the domain occupied by the fluid in the h-plane (corresponding points in the different planes are denoted by the same letters). Using expressions (4,1) and (4,2), we obtain in the upper half-plane $\operatorname{Im} w > 0$ the functional equation for $\zeta(w)$:

$$-c_1 w^{-3/2}(w-1)^{-1} = V''(w) - V'(w)\frac{\zeta''(w)}{\zeta'(w)}, \qquad c_1 > 0. \tag{4,3}$$

Here $V'(w)$ is the function defined by expression (3,4).

5. Let us investigate equation (4,3) in a neighborhood of the point $w = 0$, to which, in the ζ-plane, there corresponds the point B of contact of the free surface of the fluid with the solid boundary. Denote by β the angle between the face of the wedge and the tangent to the free-surface curve of the fluid at the point B of the ζ-plane. Then, near $u = 0$, the function $\zeta'(u)$ can be represented in the form

$$\zeta'(u) = \frac{\varphi_1(u)}{u^\gamma}, \qquad \gamma = 1 - \frac{\beta}{\pi} \quad (0 < \gamma < 1), \tag{5,1}$$

where $\varphi_1(u)$ is a bounded function. For the function $i\zeta'(u)\overline{\zeta(u)}$, near $u = 0$ we shall have

$$i\zeta'(u)\overline{\zeta(u)} = \frac{\varphi_2(u)}{u^{2\gamma-1}} \qquad (2\gamma - 1 < \gamma). \tag{5,2}$$

Under conditions (5,1), (5,2), for points w close to $w = 0$, but not lying on the real axis u, the following estimate ([9]) holds for the function $V'(w)$, defined by expression (3,4):

$$V'(w) = \frac{k}{w^\gamma} + \Phi_0(w) \qquad (k = \text{const}), \tag{5,3}$$

where $\Phi_0(w)$ is a function which is either bounded and tends to a definite limit as $w \to 0$, or is unbounded in a neighborhood of the point $w = 0$ but has order of infinity less than γ.

Using conditions (5,1) and (5,3), we obtain for the function standing on the right-hand side of equation (4,3) the following estimate near $w = 0$:

$$V''(w) - V'(w)\frac{\zeta''(w)}{\zeta'(w)} = \frac{k_1}{w^{\gamma+1}} + \Phi_1(w), \tag{5,4}$$

where the function $\Phi_1(w)$ may be unbounded in a neighborhood of $w = 0$, but its order of infinity is then less than $\gamma + 1$. On the other hand, as is seen from equation (4,3), the function (5,4) in a neighborhood of the point $w = 0$ is representable in the form

$$V''(w) - V'(w)\frac{\zeta''(w)}{\zeta'(w)} = \frac{k_2}{w^{3/2}} + \Phi_2(w), \tag{5,5}$$

where $\Phi_2(w)$ is a function unbounded in a neighborhood of $w = 0$, but having order of infinity less than $3/2$. Comparing expressions (5,4) and (5,5), po-

we obtain that $\gamma = 1/2$, whence

$$\beta = \pi/2. \tag{5,6}$$

Thus it has been proved that the free surface of the liquid forms with the solid boundary an angle equal to $\pi/2$.

6. Determining the character of the contact of the free surface of the liquid with the solid boundary makes it possible to construct the solution of the problem in the following way.

Solving the auxiliary Dirichlet problem for the upper half-plane $\operatorname{Im} w > 0$, we express $\zeta'(w)$ in terms of a new unknown function $f(u)$, representing the argument (up to the term $-\pi$) of the function $\zeta'(w)$ for $-\infty < u \leq 0$:

$$\zeta'(w) = -icw^{-1/2+\alpha/\pi}(w-1)^{-\alpha/\pi} \exp\left[\frac{w-2}{\pi}\int_{-\infty}^{0}\frac{f(u)}{u-2}\frac{du}{u-w}\right], \quad c > 0. \tag{6,1}$$

The argument of $\zeta'(w)$ for $0 \leq u < +\infty$ is known, and the function (6,1) satisfies this condition by definition. As for the function $f(u)$, it is known that it is negative for $-\infty < u \leq 0$ and tends to zero as $u \to -\infty$. In addition, from condition (5,6) we obtain the value of $f(u)$ at the point $u = 0$, namely

$$f(0) = -\alpha \tag{6,2}$$

(α is the wedge half-angle).

Now, taking into account certain properties of the function $f(u)$ as the angle of inclination of the curve of the free surface of the liquid, we construct a function satisfying the conditions written above for $f(u)$ and containing, in addition, one more parameter μ. Then from (6,1) we obtain the equation of the curve of the free surface of the liquid, and this curve will satisfy the proved condition of contact at the point B. The coordinates of the point B in the ζ-plane will be determined after the values of the constant c, the constant of integration C_*, and the value of the parameter μ have been found. To determine these constants we use the condition $\zeta(1) = -i$, the condition $\operatorname{Im}\zeta(u) = 0$ as $u \to -\infty$, and the incompressibility condition for the liquid, from which it follows that the volume of the submerged body is equal to the volume of the liquid displaced by it. After all the parameters have been determined, the complex velocity potential at any point of the region of liquid flow is obtained from expressions (3,4) and (6,1).

Institute of Mechanics
Academy of Sciences of the USSR

Received
12 VI 1963

References

1. H. Wagner, *Zs. angew. Math. u. Mech.*, **12**, No. 4, 193 (1932).
2. L. I. Sedov, *Tr. TsAGI*, issue 187 (1934).
3. L. I. Sedov, *Methods of Similarity and Dimensionality in Mechanics*, Moscow, 1957.
4. P. R. Garabedian, *Comm. Pure and Appl. Math.*, **6**, No. 2, 157 (1953).
5. S. F. Borg, *J. Eng. Mech., Divis. Proc. Am. Soc. Civ. Eng.*, **83**, No. 2 (1957).
6. E. P. Borisova, P. P. Koryavov, N. N. Moiseev, *Prikl. matem. i mekh.*, **23**, No. 2, 347 (1959).
7. M. Shiffman, D. Spenser, *Comm. Pure and Appl. Math.*, **4**, 379 (1951).
8. E. Cumberbatch, *J. Fluid Mech.*, **7**, 353 (1960).
9. F. D. Gakhov, *Boundary Value Problems*, 2nd ed., Moscow, 1963.