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Abstract

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ON INTERNAL WAVES IN AN OCEAN INHOMOGENEOUS IN DENSITY

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Progressive internal waves, which may arise in a baroclinic fluid inhomogeneous in density, were apparently first considered theoretically by Love ([1]) and Lamb ([2]). Later, Fjeldstad ([3]) gave a sufficiently general interpretation of the problem, mainly as applied to oceanographic problems, and indicated numerical methods for calculating internal waves under natural conditions.

A detailed qualitative study of the basic equation of the theory of internal waves in an inhomogeneous fluid was carried out by Eckart ([4]), who, in particular, showed that the existence of internal waves is directly due to the baroclinicity of the fluid and that the motion is vortical, while the law of conservation of vorticity is not satisfied. In the present case this also follows from the general theorems of the mechanics of a baroclinic fluid. The complex structure of the density field in the ocean, as a rule, makes a quantitative investigation difficult for many cases of practical interest. Nevertheless, below we shall indicate a sufficiently simple and at the same time typical case of density distribution in the ocean, when the solution of the problem of harmonic internal waves can be obtained in analytic form.

Let us introduce the stream function by the relations

$$u = -\frac{\partial \Psi}{\partial y}, \qquad v = \frac{\partial \Psi}{\partial x},$$

where u, v are the velocity components along the axes x, y (the y-axis is directed upward, and the motion is assumed to be two-dimensional), and let us consider harmonic waves propagating along the x-axis:

$$\Psi = \psi(y) e^{i(\omega t - kx)}.$$

Under these assumptions, from the equations of hydrodynamics we obtain (see, for example, ([4])) for $\psi(y)$ the basic equation of the linear theory of internal waves in a continuously stratified fluid:

$$\frac{d^2\psi}{dy^2} - k^2\psi + \frac{1}{\tilde{\rho}_0}\frac{d\tilde{\rho}_0}{dy}\left(\frac{d\psi}{dy} - \frac{gk^2}{\omega^2}\psi\right) = 0 \tag{1}$$

or, in a somewhat different form:

$$\frac{d}{dy}\left(\tilde{\rho}_0\frac{d\psi}{dy}\right) - \frac{k^2}{\omega^2}\left(g\frac{d\tilde{\rho}_0}{dy} + \omega^2\tilde{\rho}_0\right)\psi = 0. \tag{1a}$$

Here k is the wave number, ω the angular frequency, and g the acceleration due to gravity.

Let now the unperturbed density distribution $\tilde{\rho}_0(y)$ in the ocean have the form:

$$\tilde{\rho}_0(y) = \rho_0 = \text{const} \qquad \text{for } 0 \le y \le h,$$

$$\tilde{\rho}_0(y) = \rho_0 + \Delta\rho(1 - e^{\beta y}), \quad \Delta\rho = \rho_\infty - \rho_0 \qquad \text{for } 0 \ge y \ge -\infty. \tag{2}$$

Such a density distribution is typical for some regions of the World Ocean, in particular for the trade-wind zone of the tropical Atlantic (5), where there is an upper homogeneous layer with a thickness of the order of 100 m; then the density increases, changing substantially within the layer of the so-called "main thermocline" (600-1000 m), and becomes practically constant and equal to ρ_∞ at a depth of about 3 km. $\Delta\rho$ is a quantity of the order of 10^{-3} g/cm^3, and the order of β is 10^{-5} cm^{-1}. Taking this into account, with a high degree of accuracy we shall have:

$$\frac{1}{\tilde{\rho}_0}\frac{d\tilde{\rho}_0}{dy} = -\frac{\Delta\rho\beta e^{\beta y}}{\rho_0 + \Delta\rho(1 - e^{\beta y})} \approx -\frac{\Delta\rho\beta e^{\beta y}}{\rho_0} \equiv -se^{\beta y}. \tag{3}$$

Substituting this result into (1), we obtain:

$$\frac{d^2\psi}{dy^2} - k^2\psi - se^{\beta y}\left(\frac{d\psi}{dy} - \frac{gk^2}{i\omega^2}\psi\right) = 0.$$

Passing to the new independent variable $e^{\beta y} = \xi$, we obtain:

$$\psi'' + \left(\frac{1}{\xi} - \frac{s}{\beta}\right)\psi' + \left(\frac{gk^2 s}{\beta^2\omega^2}\frac{1}{\xi} - \frac{k^2}{\beta^2\xi^2}\right)\psi = 0.$$

Here primes denote differentiation with respect to ξ.

Since $0 \ge y \ge -\infty$ and, correspondingly, $0 \le \xi \le 1$, then everywhere $s/\beta = \Delta\rho/\rho_0 \ll 1$ ($\Delta\rho/\rho \sim 10^{-3}$), and the preceding equation may, with a high degree of accuracy, be written in the form:

$$\xi^2\psi'' + \xi\psi' + \left(q^2\xi - \frac{\nu^2}{4}\right)\psi = 0. \tag{4}$$

Here $q^2 = gk^2 s/\beta^2\omega^2,\ \nu = 2k/\beta$.

Equation (4) is integrated in cylindrical functions:

$$\psi = c_1 J_\nu(2q\xi^{1/2}) + c_2 N_\nu(2q\xi^{1/2}); \tag{5}$$

$J_\nu,\ N_\nu$ are the Bessel and Neumann functions of order ν.

If surface waves, which are of no interest in the present case, are not taken into account, then for the upper homogeneous layer the solution may be taken in the form:

$$\psi_0(y) = C_0 \operatorname{sh} k(y-h). \tag{6}$$

This solution will satisfy the condition of absence of vertical motions at the free surface, $\psi_0(h) = 0$.

If the depth of the ocean is finite, then on the horizontal bottom $y = -H$ the vertical component of velocity must also vanish. Thus,

$$\psi_0(h) = \psi(-H) = 0. \tag{7}$$

At the level $y = 0$ the following conditions must be satisfied:

$$\psi_0(0) = \psi(0), \qquad \frac{d\psi_0}{dy} = \frac{d\psi}{dy} \quad \text{for } y = 0. \tag{8}$$

Returning in (5) to the old variable y, from (5), (6), (7), (8) we obtain:

$$\psi_{0n}(y) = c_1\left[J_\nu(x_n) - \frac{J_\nu(x_n\delta)}{N_\nu(x_n\delta)}N_\nu(x_n)\right]\frac{\operatorname{sh} k(h-y)}{\operatorname{sh} kh} \quad (0 \le y \le h),$$

$$(9)$$

$$\psi_n(y) = c_1\left[J_\nu\left(x_{ne}^{\frac{\beta}{2}y}\right) - \frac{J_\nu(x_n\delta)}{N_\nu(x_n\delta)}N_\nu\left(x_{ne}^{\frac{\beta}{2}y}\right)\right] \quad (0 \ge y \ge -H),$$

where x_n are the roots of the dispersion equation:

$$\frac{\nu}{x}\left[J_\nu(x)-\frac{J_\nu(x\delta)}{N_\nu(x\delta)}\right]+\left[\frac{J_\nu(x\delta)}{N_\nu(x\delta)}N_{\nu+1}(x)-J_{\nu+1}(x)\right]+\frac{2k}{\beta}\frac{1}{x}\left[J_\nu(x)-\frac{J_\nu(x\delta)}{N_\nu(x\delta)}N_\nu(x)\right]\operatorname{cth}kh=0. \tag{10}$$

Here the notation is: $2q=x$, $\delta=e^{-\frac{\beta}{2}H}$, and the relation

$$\frac{d}{dz}J_\nu(z)=\frac{\nu}{z}J_\nu-J_{\nu+1}$$

has been used. Instead of β one may introduce the reciprocal quantity, the thickness of the "main thermocline" : $h_t=\frac{2}{\beta}$, which in each particular case must be selected from observations. Under real conditions $h_t/H\sim 2\cdot 10^{-1}$, and therefore $\delta=e^{-\frac{\beta}{2}H}=e^{-h_{tH}}\ll 1$. Determining the zeros of equation (10) in the x,ν plane is associated with great computational difficulties. However, some limiting cases, which at the same time are of the greatest practical interest, allow a comparatively simple treatment.

Using the fact that $\delta\ll 1$ and that $N_\nu(z)$ is large for small values of the argument, instead of (10) we write

$$\frac{1}{x}\left[\nu+\frac{2k}{\beta}\operatorname{cth}kh\right]J_\nu(x)-J_{\nu+1}(x)=0. \tag{11}$$

This equation can be simplified further. Assuming that kh is small, we obtain

$$J_\nu(x)=\delta' xJ_{\nu+1}(x). \tag{12}$$

Here $\delta'=h/h_t$ is also a small quantity under real conditions. When the wavelength is considerably greater than the thermocline thickness h_t, i.e. $\nu=\frac{2k}{\beta}=kh_t\to 0$, then, also taking into account the smallness of δ', instead of (12) we approximately write $J_0(x)=0$, i.e. $x=2q=\frac{2k}{\omega}\sqrt{\frac{g\Delta\rho}{\rho_0\beta}}=x_n$, where x_n are the roots of the Bessel function of zero order ($x_1=2.40$; $x_2=5.52$, etc.). For large n the asymptotic formula $x_n\simeq n\pi$ is valid. Hence, for the velocity of long waves of order n we obtain the expression:

$$c_n=\frac{\omega}{k_n}=\frac{2}{x_n}\sqrt{\frac{g\Delta\rho}{\rho_0\beta}}=\frac{\sqrt{2}}{x_n}\sqrt{\frac{\Delta\rho}{\rho_0}gh_t},\quad \text{since } h_t=\frac{2}{\beta}. \tag{13}$$

Consequently, in this limiting case the propagation speed of long waves is determined by the thermocline depth h_t. More detailed calculations show that taking into account the right-hand side of (12) leads to a certain decrease in the roots of the equation. Thus, for example, for $\delta'=0$; $\delta'=0.1$; $\delta'=0.5$ we have respectively $x_1=2.40$; $x_1=2.20$; $x_1=1.60$. In other words, the speed of internal waves increases. This is evidently connected with the fact that allowing for the nonzero thickness of the homogeneous layer, as it were, increases the "effective depth" of the ocean. On the other hand, taking into account that $\delta \neq 0$, i.e. that the ocean depth H is finite, leads to a small increase in the roots x_n.

A graphical solution of the general equation (10) for the case of long waves ($\nu \to 0$) with realistic values of the quantities: $h = 100$ m, $h_t = 1000$ m, $H = 4000$ m, i.e. $\delta' = h/h_t = 0.1$; $\delta = e^{-h_t/H} \simeq 0.02$, gives for the first three roots of this equation the following values: $x_1 = 2.75$; $x_2 = 6.00$; $x_3 = 9.35$, i.e. values that differ little from the limiting case considered above.

particular case $\delta' = \delta = 0$. This makes it possible to investigate the dispersion of internal waves on the basis of the approximate equation (12) for $\delta' = 0$. For arbitrary $\nu = kh_\tau$, instead of (17) for the first-order wave one may write:

$$c_1(\nu) = \frac{\sqrt{2}}{x_1(\nu)} \sqrt{\frac{\Delta\rho}{\rho_0}\, gh_\tau}. \tag{14}$$

Here $x_1(\nu)$ is the value of the first root of the equation $J_\nu(x) = 0$–a continuous, monotonically increasing function of ν, whose values for different ν are easily calculated using the graphs given in (6). Calculations showed that the law of variation of the velocity of internal waves as a function of wavelength is close to the analogous law for a homogeneous fluid:

$$c = \left(\frac{g\lambda}{2\pi}\,\mathrm{th}\,\frac{2\pi h}{\lambda}\right)^{1/2},$$

i.e., the velocity increases monotonically with wavelength, tending to the limiting value of the velocity of long waves. Calculation of the vertical displacements of particles, carried out by formulas (9), shows that the largest displacement amplitudes are observed not at the level $y = 0$ (where the vertical density gradient is greatest), but near the lower boundary of the thermocline, i.e., near the level $y = -h_\tau$. However, to detect this phenomenon under natural conditions, a high accuracy of observations is needed, since the gradients of hydrological characteristics (from whose oscillations internal waves can be observed) at the level of the lower boundary of the main thermocline (600–1000 m) are themselves small.

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Note: Figure translations are in progress. See original paper for figures.