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Abstract

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THE FIRST BOUNDARY-VALUE PROBLEM FOR QUASILINEAR SYSTEMS OF DIFFUSION

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In the author' s preceding work ([1]) a class of systems of equations of the mathematical theory of diffusion with a linear principal part in the parabolic operator was considered. The results of that work can be extended to more general quasilinear parabolic operators with a nonlinear principal part.

Consider a system of the form

$$\Lambda u = h(x, t, u, w),$$

$$\frac{\partial w}{\partial t} = g(x, t, u, w), \tag{1}$$

where

$$\Lambda u \equiv \sum_{i,j=1}^{n} a_{ij}\left(x, t, u, \frac{\partial u}{\partial x_k}\right)\frac{\partial^2 u}{\partial x_i \partial x_j} + f\left(x, t, u, \frac{\partial u}{\partial x_k}\right) - \frac{\partial u}{\partial t} \tag{2}$$

is a quasilinear parabolic operator.

System (1) contains two unknown functions $u(x, t)$, $w(x, t)$ and is given in a cylindrical domain $\overline{D}_T$ with boundary ∂D_T, which consists of the lateral surface S of the cylinder and its base B at $t = 0$; here B is a bounded n-dimensional domain in the space of the variables $x = (x_1, \dots, x_n)$.

We seek a solution of system (1) under the boundary conditions:

$$u(x, t) = u_0(x, t) \quad \text{on } \partial D_T; \tag{3}$$

$$w(x, t) = w_0(x) \quad \text{on } \overline{B} \text{ for } t = 0. \tag{4}$$

Problems of this type occur in the theory of nuclear reactors, in the study of heat processes in solidifying solutions, and in other questions of mathematical physics.

Problem (1), (3), (4) in the case of a linear operator Λ was studied in ([2]); in the case of one space variable, for an operator Λ quasilinear in u, it was studied by the finite-difference method in ([3]) and in the work of K. Rektorys*.

We shall assume that the following conditions are fulfilled:

A. $h(x,t,u,w)$ is a nonincreasing function of w, and $g(x,t,u,w)$ is a nondecreasing function of u; the functions h, g with respect to u and w, and the functions a_{ij} and f with respect to u, satisfy the uniform Lipschitz condition

$$|h(x,t,u_1,w_1) - h(x,t,u_2,w_2)| \leq M\left(|u_1 - u_2| + |w_1 - w_2|\right) \tag{5}$$

for $(x,t) \in \overline{D}_T$ and for all values of u, w from some bounded domain.

Essentially in the same way as in ([4]), comparison theorems are proved, the principal one of which is the following:

Theorem 1. Let $\{u_1, w_1\}$, $\{u_2, w_2\}$ be two pairs of continuous functions defined in $\overline{D}_T$. Suppose that their second derivatives with respect to x_i and first derivatives with respect to t exist, are uniformly bounded in $\overline{D}_T$, and satisfy in $\overline{D}_T$ the differential inequalities

$$\Lambda u_1 - h(x,t,u_1,w_1) \geq \Lambda u_2 - h(x,t,u_2,w_2),$$

$$\frac{\partial w_1}{\partial t} - g(x,t,u_1,w_1) \leq \frac{\partial w_2}{\partial t} - g(x,t,u_2,w_2). \tag{6}$$

* Reported at the Conference on Differential Equations and Their Applications in Prague in September 1962.

In addition, let

$$u_1 \leqslant u_2 \quad \text{on } \partial D_T, \qquad w_1 \leqslant w_2 \quad \text{on } \overline{B} \text{ when } t = 0.$$

Then

$$u_1 \leqslant u_2, \qquad w_1 \leqslant w_2, \qquad \text{in } \overline{D}_T.$$

From Theorem 1 follows the uniqueness of the solution of problem (1), (3), (4) having continuous, in the closed domain, derivatives entering the system (1).

In proving the existence theorem for the problem under consideration, the existence theorem for a classical solution of a quasilinear parabolic equation of the form

$$\sum a_{ij}\left(x,t,u,\frac{\partial u}{\partial x_k}\right)\frac{\partial^2 u}{\partial x_i \partial x_j}+A\left(x,t,u,\frac{\partial u}{\partial x_k}\right)-\frac{\partial u}{\partial t}=0. \tag{7}$$

will be used. For example, for definiteness one may consider the operator Λ in the form

$$\Lambda_1 u \equiv \frac{da_i\left(x,t,u,\partial u/\partial x_k\right)}{dx_i}+a\left(x,t,u,\frac{\partial u}{\partial x_k}\right)-\frac{\partial u}{\partial t} \tag{8}$$

and use the results of O. A. Ladyzhenskaya and N. N. Ural' tseva ([5]) on the first boundary-value problem for quasilinear parabolic equations.

Let the following conditions be satisfied:

B. Let $\Psi(x,t)$ be an extension into the interior of the domain of the boundary function $u_0(x,t)$, and let $\Psi(x,t)$ have second derivatives with respect to x_i and first derivatives with respect to t, satisfying the Hölder condition with exponent β $(0<\beta<1)$ in x and the Lipschitz condition in t. In this case, under those conditions on the coefficients which will be formulated below, the boundary condition $u_0(x,t)$, without loss of generality, can be reduced to zero. Let the compatibility conditions of zero and first order for the boundary function $u_0(x,t)$ also be satisfied for $x \in S$ when $t=0$.

C. Let the lateral boundary $S \in C_2^{\beta}$, i.e., have second derivatives with respect to x_i, satisfying the Hölder condition with exponent β.

D. Let the coefficients of the operator Λ_1 in (8) satisfy the following conditions (cf. (5^6)):

1) For $(x,t)\in \overline{D}_T$ and arbitrary $u(x,t)$, the inequalities

$$u\left[\frac{\partial a_i(x,t,u,0)}{\partial x_i}+a(x,t,u,0)\right]\geqslant -b_1u^2-b_2, \qquad b_i\geqslant 0; \tag{9}$$

$$\left.\frac{\partial a_i(x,t,u,p_k)}{\partial p_j}\xi_i\xi_j\right|_{p=0}\geqslant 0, \tag{10}$$

are fulfilled, where

$$p_k=\partial u/\partial x_k, \qquad p=\left(\sum_{k=1}^{n}p_k^2\right)^{1/2}.$$

2) In each finite part of the domain $\{(x,t) \in \overline{D}_T,\ |u| < \infty\}$ and for arbitrary p_k, the functions $a_i(x,t,u,p_k)$ and $a(x,t,u,p_k)$ are continuous, a_i are differentiable with respect to x_k, u, p_k, and they are subject to the inequalities

$$\frac{\partial a_i(x,t,u,p_k)}{\partial p_j}\xi_i\xi_j \geq \nu(|u|)\sum_{i=1}^{n}\xi_i^2;$$

$$\sum_{i=1}^{n}\left(|a_i| + \left|\frac{\partial a_i}{\partial u}\right| + \sum_{j=1}^{n}\left|\frac{\partial a_i}{\partial x_j}\right|\right)(p+1)+$$

$$+\sum_{i,j=1}^{n}\left|\frac{\partial a_i}{\partial p_j}\right|(p+1)^2 + |a| \leq \mu(|u|)(p+1)^2. \tag{11}$$

3) In every finite part of the domain $\{(x,t) \in \overline{D}_T,\ |u| < \infty,\ |p| < \infty\}$ the functions $a_i, a, \partial a_i/\partial p_j, \partial a_i/\partial u, \partial a_i/\partial x_j$ are continuous, and satisfy, with respect to x_k, t, u, p_k, the Hölder condition with exponents $\beta, \beta/2, \beta, \beta$, respectively; moreover, a_i, a, h with respect to t, and $a, \partial a_i/\partial p_j, \partial a_i/\partial u, \partial a_i/\partial x_j$ with respect to u, satisfy the Lipschitz condition, the function $a(x,t,u,p_k)$ is differentiable with respect to u, p_k, and the constants in the Lipschitz conditions and the quantities $|\partial a/\partial u|$, $|\partial a/\partial p_k|$ are bounded by some constant M.

Theorem 2. *Suppose that conditions A, B, C, D are fulfilled; the function $w_0(x)$ satisfies the Hölder condition with exponent $\beta > 0$; the functions $h(x,t,u,w)$, $g(x,t,u,w)$ satisfy the Hölder condition with exponent β in x_i; the function g satisfies the Hölder condition with exponent $\beta/2$ in t for $(x,t) \in \overline{D}_T$ and all bounded values of u, w.*

Suppose, moreover, that there exist two pairs of functions $\{u', w'\}, \{u'', w''\}$, continuous in (x,t) and satisfying the Hölder condition with exponents $\beta, \beta/2$, respectively, and possessing continuous and bounded second derivatives with respect to x_i and first derivatives with respect to t, which satisfy the system of inequalities:

$$\begin{gathered}
\Lambda_1(u') - h(x,t,u',w') \geq 0 \geq \Lambda_1(u'') - h(x,t,u'',w''),\\
\frac{\partial w'}{\partial t} - g(x,t,u',w') \leq 0 \leq \frac{\partial w''}{\partial t} - g(x,t,u'',w''),\\
u'(x,t) \leq u_0(x,t) \leq u''(x,t) \quad \text{on } \partial D_T,\\
w'(x,0) \leq w_0(x) \leq w''(x,0) \quad \text{for } t=0.
\end{gathered} \tag{12}$$

Then there exists a solution of the system (1), (3), (4) with operator Λ_1, belonging to the classes

$$u \in C_{2,1}^{\gamma,\gamma/2}(\overline{D}_T); \qquad w,\ \partial w/\partial t \in C^{\gamma,\gamma/2}(\overline{D}_T)$$

with some γ $(0 < \gamma \leq \beta)$.

This theorem is proved by the method of successive approximations. As the zeroth approximation the functions u'', w'' are taken, and the successive approximations are found from the system

$$\Lambda_1 u_n - M u_n = h(x, t, u_{n-1}, w_{n-1}) - M u_{n-1}; \tag{13}$$

$$\frac{\partial w_n}{\partial t} + M w_n = g(x, t, u_{n-1}, w_{n-1}) + M w_{n-1}; \tag{14}$$

$$u_n(x, t) = u_0(x, t) \quad \text{on } \partial D_T; \tag{15}$$

$$w_n(x, t) = w_0(x, 0) \quad \text{for } t = 0, \tag{16}$$

where M is the constant taken from condition A.

Each equation (13), (14), with the corresponding boundary conditions (15) and (16), is uniquely solvable; (13), (15) by virtue of Theorem 2 of paper ([56]).

Next, the convergence at every point of the domain $\overline{D}_T$ of the sequences $\{u_n\}, \{w_n\}$ is proved; moreover, the inequalities

$$u' \leq u_n \leq u_{n-1} \leq u'',$$

$$w' \leq w_n \leq w_{n-1} \leq w'' \tag{17}$$

hold.

Thus, the limit of the sequences $\{u_n\}, \{w_n\}$ exists and defines functions $u(x, t), w(x, t)$ in $\overline{D}_T$. On the basis of (17) and Theorem 1 of paper ([56]), the following estimate holds in the metric $C^{1+\alpha}$:

$$|u_n|_\alpha + \left|\frac{\partial u_n}{\partial x_k}\right|_\alpha < c \tag{18}$$

for some $\alpha > 0$, where the constant c does not depend on n. Therefore from the sequence $\{u_n\}$ one can select a subsequence $\{u_{n_k}\}$ converging in the metric $C^{1+\alpha_1}$, where $0 < \alpha_1 < \alpha < 1$. Passing to the limit along this sub-

of a sequence; we obtain that $u_{n_k} \to u(x, t)$, $\partial u_{n_k}/\partial x_i \to \partial u/\partial x_i$ uniformly, and we have Hölder continuity of the functions $u(x, t)$, $\partial u/\partial x_i$ with exponent α_1, where $0 < \alpha_1 < \alpha < 1$.

Likewise, as is proved in [4], w, $\partial w/\partial t$ satisfy the Hölder condition with exponent β in $\overline{D}_T$.

It is then proved that $\{u, w\}$ is a solution of the posed problem and

$$u \in C_{2,1}^{\gamma,\gamma/2}(\overline{D}_T); \qquad w,\ \frac{\partial w}{\partial t} \in C^{\gamma,\gamma/2}(\overline{D}_T)$$

for some γ $(0 < \gamma \leqslant \beta)$.

The operators Λ may also be taken in nondivergence form, and also with arbitrary growth in p, but satisfying conditions such that the corresponding boundary-value problem is solvable for it in the classical sense and a priori estimates of type (18) hold. In particular, one may take an operator Λ satisfying the conditions of Theorem 5 of [5] or the conditions of Theorem 10 of [5].

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Note: Figure translations are in progress. See original paper for figures.