CYBERNETICS AND CONTROL THEORY

1963

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Abstract

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CYBERNETICS AND CONTROL THEORY

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THE COMPLEXITY OF CONTACT CIRCUITS REALIZING ONE FUNCTION OF THE ALGEBRA OF LOGIC

(Presented by Academician S. L. Sobolev on 18 II 1963)

The problem of finding a minimal circuit of a specified kind realizing a given function of the algebra of logic is of practical interest. For its solution it is necessary to find a satisfactory lower bound for the complexity of an arbitrary circuit realizing the given function.

According to Shannon (1), for almost all functions of the algebra of logic in n arguments, the number of contacts in a minimal circuit depends exponentially on n. Nevertheless, so far only a few examples of functions have been constructed for which the complexity of the minimal circuit depends nonlinearly on n; one may mention the works of A. A. Markov (2), B. A. Subbotovskaya (3), and O. B. Lupanov (4). In the work of O. B. Lupanov (4) it is proved that an arbitrary contact circuit containing no break contacts and realizing the function

$$f_0(x_1, \dots, x_n) = \bigvee_{1 \le i < j \le n} x_i x_j$$

has not fewer than $C_1 \dfrac{n \log_2 n}{\log_2 \log_2 n}$ contacts, where C_1 is some positive constant.

The purpose of the present note is to prove the following two assertions:

1. An arbitrary contact circuit containing no break contacts and realizing the function $f_0(x_1, \dots, x_n)$ has not fewer than $C_2 n \log_2 n$ contacts, where C_2 is some positive constant (Theorem 2).
2. There exists a minimal series-parallel contact circuit realizing $f_0(x_1, \dots, x_n)$ and containing no break contacts (Corollary of Lemma 1).

Consequently, an arbitrary series-parallel contact circuit realizing $f_0(x_1, \dots, x_n)$ has not fewer than $C_2 n \log_2 n$ contacts.

Let us note that these lower bounds differ only by a factor not depending on n from the upper bound, given in the work of V. K. Korobkov (5), for the complexity of the minimal series-parallel circuit realizing $f_0(x_1, \dots, x_n)$.

A formula in the basis $\vee, \cdot, -$ will henceforth be called simply a formula. The number of variable symbols entering into a formula $G(x_1, \dots, x_n)$ will be called its complexity and denoted by $L(G(x_1, \dots, x_n))$. We agree that the empty formula has zero complexity and realizes the function identically equal to zero. If $g(x_1, \dots, x_n)$ is a function of the algebra of logic, then by $L(g(x_1, \dots, x_n))$ we shall denote the complexity of the simplest of the formulas realizing it. It is known that there is an isomorphism between formulas and series-parallel circuits.

Let $g(x_1, \dots, x_n) = 0$ if $x_1 + \dots + x_n = 1^*$. By $H(g(x_1, \dots, x_n))$ we shall denote the set of all functions of the algebra of logic $g^\alpha(x_1, \dots, x_n)$ such that

A. $g^\alpha(x_1, \dots, x_n) = 0$, if $x_1 + \dots + x_n = 1$.

B. $g^\alpha(x_1, \dots, x_n) \geq g(x_1, \dots, x_n)$, if $x_1 + \dots + x_n \neq 0$.

* The sign $+$ denotes arithmetic addition.

Let

$$L(H(g(x_1, \dots, x_n))) = \min L(g^\alpha(x_1, \dots, x_n)), \quad g^\alpha(x_1, \dots, x_n) \in H(g(x_1, \dots, x_n)).$$

We shall denote by $H^*(g(x_1, \dots, x_n))$ the set of all those functions $g^\alpha(x_1, \dots, x_n) \in H(g(x_1, \dots, x_n))$ for which the condition

$$L(g^\alpha(x_1, \dots, x_n)) = L(H(g(x_1, \dots, x_n)))$$

is satisfied. $H^*(g(x_1, \dots, x_n))$ is nonempty for every function $g(x_1, \dots, x_n)$.

Lemma 1. *Let $g(x_1, \dots, x_n) = 0$ if $x_1 + \dots + x_n = 1$. Then there exists a function $g^*(x_1, \dots, x_n) \in H^*(g(x_1, \dots, x_n))$, one of whose minimal formulas has the form*

$$\bigvee_{i=1}^{T} \left(\left(\bigvee_{l \in A_i} x_l \right) \cdot \left(\bigvee_{l' \in A'_i} x_{l'} \right) \right). \tag{1}$$

Here A_i and A'_i are certain subsets of the set $(1, 2, \dots, n)$, $A_i \cap A'_i = \emptyset$, and, if one of the sets A_i and A'_i is empty, then the other is empty as well; $i = 1, 2, \dots, T$, $T \geq 1$.*

Proof. We prove the assertion by induction on the number of variables of the function $g(x_1, \dots, x_n)$. For $n = 1$ the lemma is obvious. Suppose the lemma has been proved for functions having $n-1$ arguments. Let $g(x_1, \dots, x_n) \not\equiv 0$, $g(x_1, \dots, x_n) = 0$ when $x_1 + \dots + x_n = 1$, $g^1(x_1, \dots, x_n) \in H^*(g(x_1, \dots, x_n))$, and let $G^1(x_1, \dots, x_n)$ be some minimal formula for the function $g^1(x_1, \dots, x_n)$.

We may assume that every subformula different from a variable enters into $G^1(x_1, \dots, x_n)$ without negation.

$$G^1(x_1, \dots, x_n) = \bigvee_{i=1}^{T} G_i(x_1, \dots, x_n) \cdot G'_i(x_1, \dots, x_n), \tag{2}$$

where $T \geq 1$, and $G_i(x_1, \dots, x_n)$ and $G'_i(x_1, \dots, x_n)$, $i = 1, \dots, T$, are certain formulas realizing the functions not identically equal to one, $g_i(x_1, \dots, x_n)$ and $g'_i(x_1, \dots, x_n)$.

We shall call a variable x_i, $1 \leq i \leq n$, **distinguished for** $g_1(x_1, \dots, x_n)$ if $g_1(x_1, \dots, x_n)$ essentially depends on x_i and $g_1(0, \dots, 0, 1, 0, \dots, 0) = 1$ (the one is in the i-th position). It follows from A that the sets of variables distinguished for $g_1(x_1, \dots, x_n)$ and $g'_1(x_1, \dots, x_n)$ do not intersect. Suppose, without loss of generality, that $x_1, \dots, x_k$ and $x_{k+1}, \dots, x_{k+l}$ are all the variables distinguished respectively for $g_1(x_1, \dots, x_n)$ and $g'_1(x_1, \dots, x_n)$, $0 \leq k \leq n$, $0 \leq k + l \leq n$. It is easy to verify that the function $g^2(x_1, \dots, x_n)$, realized by the formula $G^2(x_1, \dots, x_n)$,

$$\begin{aligned} G^2(x_1, \dots, x_n) = {} & (x_1 \vee \dots \vee x_k \vee G_1(0, \dots, 0, x_{k+1}, \dots, x_n)) \\ & \cdot (x_{k+1} \vee \dots \vee x_{k+l} \vee G'_1(x_1, \dots, x_k, 0, \dots, 0, x_{k+l+1}, \dots, x_n)) \\ & \vee \bigvee_{i=2}^{T} G_i(x_1, \dots, x_n) \cdot G'_i(x_1, \dots, x_n), \end{aligned} \tag{3}$$

again belongs to $H^*(g(x_1, \dots, x_n))$ and $L(g^2(x_1, \dots, x_n)) = L(g^2(x_1, \dots, x_n))$.

Let $g_1(0, \dots, 0, x_{k+1}, \dots, x_n) = \varphi(x_{i_1}, \dots, x_{i_\beta})$, where $x_{i_1}, \dots, x_{i_\beta}$ are all essential variables of the function $g_1(0, \dots, 0, x_{k+1}, \dots, x_n)$. It is easy to see that $\beta < n$. Since $x_1, \dots, x_k$ are all distinguished variables of the function $g_1(x_1, \dots, x_n)$, we have $\varphi(x_{i_1}, \dots, x_{i_\beta}) = 0$ when $x_{i_1} + \dots + x_{i_\beta} = 1$. Therefore, by the induction hypothesis, there exists a function $\varphi^*(x_{i_1}, \dots, x_{i_\beta}) \in H^*(\varphi(x_{i_1}, \dots, x_{i_\beta}))$, having a minimal formula $\Phi^*(x_{i_1}, \dots, x_{i_\beta})$ of the form (1). An analogous function $\varphi'^*(x_{j_1}, \dots, x_{j_\gamma})$ exists for

$$\varphi'(x_{j_1}, \dots, x_{j_\gamma}) = g'(x_1, \dots, x_k, 0, \dots, 0, x_{k+l+1}, \dots, x_n), \quad 0 \leq \gamma < n.$$

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* The empty formula has the form (1). The function $g^*(x_1, \dots, x_n)$ realized by a formula of the form (1) is monotone, and $g^*(x_1, \dots, x_n) = 0$ when $x_1 + \dots + x_n \leq 1$.

Consider the function $g^3(x_1, \dots, x_n)$ realized by the formula $G^3(x_1, \dots, x_n)$:

$$\begin{aligned} G^3(x_1,\dots,x_n) = (x_1 \vee \dots \vee x_k)\cdot(x_{k+1} \vee \dots \vee x_{k+l}) \vee \Phi^*(x_{i_1},\dots,x_{i_\beta}) \vee \\ \vee \Phi'^*(x_{j_1},\dots,x_{j_\gamma}) \vee \bigvee_{i=2}^{r} G_i(x_1,\dots,x_n)\, G'_i(x_1,\dots,x_n)^*. \end{aligned} \tag{4}$$

It can be shown that $g^3(x_1,\dots,x_n) \in H^*(g(x_1,\dots,x_n))$, and $L(G^3(x_1,\dots,x_n)) = L(g^3(x_1,\dots,x_n))$. Transforming the terms $G_i(x_1,\dots,x_n)\cdot G'_i(x_1,\dots,x_n)$, for $i>1$, in the same way as the first, we obtain a function $g^*(x_1,\dots,x_n) \in H^*(g(x_1,\dots,x_n))$, where the minimal formula for $g^*(x_1,\dots,x_n)$ will have the form (1). The lemma is proved.

Corollary. For the function $f_0(x_1,\dots,x_n)$ there exists a minimal formula of the form (1).

Denote by $m_i(F(x_1,\dots,x_n))$ the number of occurrences of the variable x_i in the formula $F(x_1,\dots,x_n)$, $1 \le i \le n$.

Lemma 2. For the function $f_0(x_1,\dots,x_n)$ there exists a minimal formula $F_0(x_1,\dots,x_n)$ of the form (1) such that

$$\left|m_i(F_0(x_1,\dots,x_n)) - m_j(F_0(x_1,\dots,x_n))\right| \le 1, \qquad 1 \le i \le n, \quad 1 \le j \le n.$$

Theorem 1. The function $L(f_0(x_1,\dots,x_n))$ satisfies the inequality

$$L(f_0(x_1,\dots,x_n)) \ge C_2 n \log_2 n,$$

where $C_2 = 1/4$.

Proof. We prove the theorem by induction. Let in the formula $F_0(x_1,\dots,x_n)$, whose existence is asserted in Lemma 2, $n-p$ variables occur the maximal number m of times, and the remaining p variables $m-1$ times, $0 \le p \le n$. Take an arbitrary disjunctive term of the formula $F_0(x_1,\dots,x_n)$, for example the first:

$$\left(\bigvee_{l\in A_1} x_l\right)\cdot\left(\bigvee_{l'\in A'_1} x_{l'}\right).$$

Let the sets A_1 and A'_1 consist respectively of a and a' elements. We shall assume that $a \le a'$. Suppose that among the variables x_l, $l \in A_1$, there are b_1 variables occurring in $F_0(x_1,\dots,x_n)$ m times, and $b_2 = a - b_1$ variables occurring $m-1$ times. Put $x_l = 0$ if $l \in A_1$. We obtain:

$$L(n) \ge a' + b_1 m + b_2(m-1) + L(n-a), \tag{5}$$

where $L(n) = L(f_0(x_1, \dots, x_n))$.

Taking into account that $m = \dfrac{L(n) + p}{n}$ and using the induction hypothesis, after simple transformations we obtain

$$L(n) \geq \frac{n(a' - b_2) + ap}{n - a} - \frac{a}{2} + \frac{1}{4} n \log_2 n. \tag{6}$$

For $p \geq n/2$ the assertion of the theorem follows directly from (6). For $p \leq n/2$ we use the fact that there is some term (let it be the first) in which $b_2/(a + a') \leq p/n$. From this inequality and from (6) Theorem 1 follows. The theorem is proved.

Let now S be an arbitrary contact circuit of closing contacts realizing $f_0(x_1, \dots, x_n)$, and let π_1 and π_2 be its poles. If some vertex α is connected with π_1 by a bundle of parallel contacts, then among the contacts of this bundle and the other contacts incident with α there are no identical ones (see (^{4})). If α is not connected by a contact with π_1, then identify it with π_2. Since every chain issuing from π_1 passes through two different contacts, the new circuit will again realize $f_0(x_1, \dots, x_n)$ and will be no more complex than the original. Thus, we may assume that there exists such a minimal circuit of closing contacts realizing $f_0(x_1, \dots, x_n)$ in which every vertex is adjacent to both poles.

Theorem 2. A contact circuit containing only make contacts and realizing $f_0(x_1, \dots, x_n)$ contains at least $C_3 n \log_2 n$ contacts, where $C_3 = \text{const}$.

Proof. In the circuit S, replace every bridge contact (here, one not incident to any pole) a by two, as shown in Fig. 1. We obtain a series-parallel circuit S_1 with conductivity function $f_1(x_1, \dots, x_n)$. The conjunction corresponding to a chain in S passing through an arbitrary bridge contact has the form $b \cdot c \cdot \dots$, where b is incident to π_1, and c is some bridge contact incident to b. This conjunction corresponds in S_1 to a chain with conductivity $b \cdot c$. The remaining chains in S and S_1 are identical. Therefore

$$f_0(x_1, \dots, x_n) \leq f_1(x_1, \dots, x_n).$$

On the other hand, in S_1 every chain passes through two different contacts; therefore

$$f_1(x_1, \dots, x_n) \leq f_0(x_1, \dots, x_n).$$

Thus the series-parallel circuit S_1 realizes $f_0(x_1, \dots, x_n)$. Since S_1 is at most twice as complex as S, Theorem 2 follows from Theorem 1 if we put $C_3 = \frac{1}{2} C_2$.

Fig. 1

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* If $k = 0$ or $l = 0$, then the term $(x_1 \vee \dots \vee x_k) \cdot (x_{k+1} \vee \dots \vee x_{k+l})$ is absent.

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Received
14 II 1963

CITED LITERATURE

1. C. E. Shannon, Bell Syst. Techn. J., **28**, No. 1, 59 (1949).
2. A. A. Markov, Abstracts of Reports at the All-Union Conference on the Theory of Relay Devices, Moscow, 1957.
3. B. A. Subbotovskaya, DAN, **136**, No. 3, 553 (1961).
4. B. L. Ulyanov, DAN, **144**, No. 6, 1245 (1962).
5. V. K. Korobkov, DAN, **109**, No. 2, 260 (1956).

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