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Abstract

Full Text

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On Lie Algebras with a Regular Automorphism

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1.

An automorphism Φ acting on an algebra $\mathfrak{L}$ is called regular if $\Phi(x) \neq x$ for all $x \neq 0$, $x \in \mathfrak{L}$. One of the aims of this note is the proof of the following assertion, nontrivial for an integer $s > 1$.

Theorem A. *A finite-dimensional Lie algebra over a field of characteristic $p > 0$, admitting a regular automorphism of period $m = q^s < p$ (q prime), is solvable.*

It is well known ([1,2]) that a Lie algebra with a regular automorphism of prime period q is nilpotent; moreover, Higman ([2]) even showed that for its nilpotency class there exists an upper bound depending (in a still unknown way) only on q. On the other hand, a Lie algebra with a regular automorphism of arbitrary period may fail to be nilpotent. It seems a very plausible hypothesis that every finite-dimensional Lie algebra with a regular automorphism (not necessarily of finite period) is solvable. If the ground field has characteristic zero, then this is indeed so ([3]); see also below the corollary of Lemma 2. As for Lie algebras of finite characteristic, as far as the authors know, there are no general assertions on this question. The reason for such a sharp difference in the level of the results obtained lies in the fact that the problem under consideration essentially reduces to the study of the so-called degenerate simple algebras ([4]), i.e., to a situation characteristic only of the case $p > 0$.

2.

We next introduce the necessary notation and terminology. The ground field $\mathfrak{K}$ is assumed algebraically closed (which, however, is inessential). The Lie algebra $\mathfrak{L}$, considered as a vector space over $\mathfrak{K}$, is represented in the form $\mathfrak{L} = \sum \mathfrak{L}_\lambda$, where $\mathfrak{L}_\lambda$ is the root subspace in $\mathfrak{L}$ corresponding to the characteristic root λ of the linear endomorphism Φ. The degree n of the endomorphism Φ will mean the degree of the corresponding minimal polynomial

$$f(t) = (t - \lambda_1)^{s_1} \cdots (t - \lambda_r)^{s_r},$$

$$n = s_1 + \cdots + s_r$$

(λ_i all distinct). Without loss of generality the endomorphism Φ may be assumed semisimple, i.e.

$$f(t) = (t - \lambda_1) \cdots (t - \lambda_n),$$

$s_i = 1$, $r = n$, and $\Phi(x) = \lambda x$ for all $x \in \mathfrak{L}_\lambda$. The fact that Φ is a regular automorphism of the algebra $\mathfrak{L}$ is expressed by the conditions $\lambda_i \neq 0, 1$ and

$$[\mathfrak{L}_{\lambda_i} \mathfrak{L}_{\lambda_j}] \subseteq \mathfrak{L}_{\lambda_i \lambda_j}.$$

The adjoint endomorphism $y \to [yx]$ of the vector space $\mathfrak{L}$, corresponding to an element $x \in \mathfrak{L}$, will be denoted by the capital letter X. An element x has nilpotency index h if $X^h = 0$, $X^{h-1} \neq 0$. A simple algebra $\mathfrak{L}$ is called degenerate if the fundamental bilinear form

$$B(x, y) = \operatorname{Tr} XY$$

is identically equal to zero. We shall also say that a simple algebra $\mathfrak{L}$ has strong degeneration if $C^2 = 0$ for some element $c \neq 0$ of $\mathfrak{L}$ (an algebra with condition (*) in the article ([4])). By definition, $\mathfrak{L}(m)$ is the set of all those elements $c \neq 0$ of $\mathfrak{L}$ which satisfy identically

$$C X^{iC} = 0, \quad i = 0, 1, \dots, 2m - 1, \quad x \in \mathfrak{L}.$$

In what follows we shall also call an element $x \in \mathfrak{L}$ homogeneous if $x \in \mathfrak{L}_\lambda$ for some λ.

3. **Lemma 1.** If $\mathfrak{L}$ is a nonsolvable Lie algebra with a regular automorphism Φ of degree n (of finite period m), then there also exists a simple Lie algebra with a regular automorphism whose degree does not exceed n (respectively, of period m', $m' \mid m$).

The proof of this lemma is close to the proof of Patterson' s theorem 3 ([3]). In the case $p > 0$ it is necessary to make only minor changes.

Lemma 2. A simple Lie algebra $\mathfrak{L}$ with a regular automorphism Φ is degenerate. At least one root subspace $\mathfrak{L}_\lambda$ consists entirely of elements whose adjoint endomorphisms are nilpotent. The nilpotency index of any of them does not exceed the degree n of the automorphism Φ.

Proof. If $z \in \mathfrak{L}_\lambda$ and $a \in \mathfrak{L}_\mu$, then zA^h is either equal to 0 or is contained in $\mathfrak{L}_{\lambda\mu^h}$. Surely $A^n = 0$ when μ is not a root of some degree of 1. Therefore it is necessary to consider only the following case:

$$\lambda_1 = \varepsilon^{m_1}, \dots, \lambda_n = \varepsilon^{m_n}, \quad \varepsilon^m = 1$$

(ε is a primitive root),

$$0 < m_1 < m_2 < \dots < m_n < m.$$

Let $a \in \mathfrak{L}_{\lambda_1}$, $z \in \mathfrak{L}_{\lambda_i}$, $m_i + h_i m_1 = m + r_i$, $0 \leq r_i < m_1$.

From the regularity of the automorphism and the minimality of m_1 among the m_i it follows that $\mathfrak{L}_{\varepsilon^{r_i}} = 0$. Thus, $[za^{h_i}] = 0$. If $h_i > n$ and $[za^n] \neq 0$, then more than n distinct root subspaces are obtained $(m_i + nm_1 < m)$. Therefore $zA^n = 0$ and $A^n = 0$. $\operatorname{Tr} AZ = 0$ when $m_1 + m_i \not\equiv 0 \pmod{m}$, for an obvious reason. If $m_1 + m_i \equiv 0 \pmod{m}$, then $[az] \in \mathfrak{L}_1 = 0$, $(AZ)^n = A^n Z^n = 0$, and again $\operatorname{Tr} AZ = 0$. Thus the fundamental bilinear form is degenerate, and more precisely is identically equal to zero.

Corollary. A finite-dimensional Lie algebra over a field of characteristic zero is solvable if a regular automorphism acts on it.

Lemma 3. Under the conditions of Lemma 2 the algebra $\mathfrak{L}$ is strongly degenerate, if $n < p$.

Indeed, according to Lemma 2, in the algebra $\mathfrak{L}$ there exists a homogeneous element a of nilpotency index $h \leq n$. Starting from it and using the construction of the proof of Theorem 2 from ([5]), one can find an element $b \in \mathfrak{L}_\lambda$, $B^3 = 0$. Next we argue as follows. We regard the element b as chosen so that

$$\dim[\mathfrak{L}b^2] = \min\{\dim[\mathfrak{L}x^2]\},$$

where x runs over the whole set of homogeneous elements of nilpotency index 3. If $[ub^2] \neq 0$, $u \in \mathfrak{L}_\mu$, then $[UB^2]^2 = B^2U^2B^2$, $[UB^2]^3 = 0$ (see ([5])) and $[\mathfrak{L}[ub^2]^2] \subset [\mathfrak{L}h^2]$, while the choice of the element b leads to the equality $[\mathfrak{L}[ub^2]^2] = [\mathfrak{L}b^2]$. In particular, $[ub^2] = [v[ub^2]^2]$ for some element $v \in \mathfrak{L}_\nu$. However, from this relation it follows that $\lambda^2\mu\nu = 1$, i.e. $[ub^2] = 0$. The contradiction obtained shows that $[\mathfrak{L}b^2] = 0$, or $b \in \mathfrak{L}(1)$.

Lemma 4. In a simple Lie algebra $\mathfrak{L}$ with a regular automorphism Φ of degree $n < p$ there exists a homogeneous element $c \in \mathfrak{L}\left(\frac{p-3}{2}\right)$.

Proof. The nonemptiness of the set $\mathfrak{L}\left(\frac{p-3}{2}\right)$ follows from Lemma 3 and Theorem 1 ([4]). Thus it remains only to establish the existence in $\mathfrak{L}\left(\frac{p-3}{2}\right)$ of a homogeneous element. A small modification of the arguments of § 1 of ([5]) permits this to be done under the condition that there exists a homogeneous element in the set $\mathfrak{L}(2)$. We shall prove the latter for $p \geq 11$, omitting the simple direct computations for $p = 7$. It is known (([4]), Theorem 5) that the subalgebra $\mathfrak{C}$ generated by the set $\mathfrak{L}(1)$ is nilpotent, and the intersection $\mathfrak{z}(\mathfrak{C}) \cap \mathfrak{L}\left(\frac{p-3}{2}\right)$, where $\mathfrak{z}(\mathfrak{C})$ –

the center in $\mathfrak{G}$ is nonempty. Let

$$c' = c_1 + c_2 + \dots + c_n \in \mathfrak{z}(\mathfrak{G}) \cap \Omega\left(\frac{p-3}{2}\right),$$

$c_i \in \Omega_{\lambda_i}$, $c_k \neq 0$, for some k. From the linear system

$$\Phi^i(c') = \lambda_1^i c_1 + \lambda_2^i c_2 + \dots + \lambda_n^i c_n, \qquad i = 0, 1, \dots, n-1,$$

we find

$$c_k = \sum \alpha_i \Phi^i(c').$$

Denoting by $\Phi^i(C')$ the adjoint endomorphism corresponding to the element $\Phi^i(c')$, consider the expression

$$C_k X^l C_k = \sum \alpha_i \alpha_j \Phi^i(C') X^l \Phi^j(C'), \qquad x \in \Omega, \quad l = 0, 1, 2.$$

Obviously,

$$[y c_k x^l c_k] = \sum \alpha_i \alpha_j \Phi^i([y' c' x^l \Phi^{j-i}(c')]), \qquad y' = \Phi^{-i}(y), \quad x' = \Phi^{-i}(x).$$

But

$$[Y' C' X'^l]^2 = \sum_{\nu_i \le 2l+2 \le 6 < p-4} \cdots C' Z^{\nu_i} C' \cdots = 0,$$

i.e. $[y' c' x'^l] \in \Omega(1)$. And since $\Phi^{j-i}(c') \in \mathfrak{z}(\mathfrak{G})$, it follows that

$$[y' c' x'^l \Phi^{j-i}(c')] = 0.$$

Consequently,

$$C_k X^l C_k = 0,$$

or $c_k \in \Omega(2)$.

In the proof of Theorem 1, besides Lemmas 1-4, the following number-theoretic fact plays an essential role.

Lemma 5. Let m be any natural number and

$$\Sigma = \{\underbrace{\sigma_1, \dots, \sigma_1}_{m_1};\ \dots;\ \underbrace{\sigma_r, \dots, \sigma_r}_{m_r}\}, \qquad m_1 + m_2 + \dots + m_r = m - 1,$$

be a certain set of residues modulo m, where $\sigma_i \ne \sigma_j$ for $i \ne j$ and $(\sigma_i, m) = 1$, $i = 1, \dots, r$. Then for an arbitrary nonzero residue σ the congruence

$$g_1 \sigma_1 + \dots + g_r \sigma_r \equiv \sigma \pmod m$$

is solvable, and the components g_i of the solution satisfy the inequalities

$$0 \le g_i \le m_i, \qquad i = 1, \dots, r.$$

Strictly speaking, we use a slightly stronger assertion, which, because of its unwieldiness, is not formulated here.

4. In connection with the remark in §1 on Higman' s result, it may be useful to indicate the relation between the nilpotency class c of an algebra Ω with a regular automorphism Φ of prime period q and the length d of its derived series.

Theorem B.

$$c < \frac{(q-1)^d - 1}{q-2} + 1.$$

For the proof it is necessary to use the idea proposed by Higman[6], who established an analogous relation between the numbers c and d in Engel Lie rings. The decisive point is the proof of the inclusion

$$\Omega^{(q-1)s+2} \subseteq \mathfrak{M}^{s+1},$$

where $\mathfrak{M} = [\Omega\Omega]$. We are given the decomposition

$$\Omega = \sum_{i=1}^{q-1} L_i, \qquad L_i = \Omega_{\varepsilon^i}, \quad \varepsilon^q = 1, \quad [L_i L_j] \subseteq L_{i+j}, \quad L_0 = 0.$$

Let

$$\Omega^{(q-1)(s-1)+2} \subseteq \mathfrak{M}^s.$$

If $u \in \mathfrak{M}^s \cap L_k$, $v_j \in L_{k_j}$, $j = 1, 2, \dots, q-1$, then the element

$$[uv_1 \cdots v_{q-1}] \in \mathfrak{M}^{s+1}.$$

Indeed,

$$[v_\nu v_\mu] \in \mathfrak{M}, \qquad [uv_1 \cdots [v_\nu v_\mu] \cdots] \in \mathfrak{M}^{s+1}.$$

Therefore, modulo terms from $\mathfrak{M}^{s+1}$, we may replace $[uv_1 \cdots v_{q-1}]$ by the element

$$[uv_{\pi 1} \cdots v_{\pi(q-1)}],$$

where π is any permutation of the indices $1, 2, \dots, q-1$. According to Lemma 5, with a suitable choice of π, the congruence

$$k + k_{\pi 1} + \cdots + k_{\pi t} \equiv 0 \pmod q, \qquad t \leq q-1,$$

will be satisfied, i.e.

$$[uv_{\pi 1} \cdots v_{\pi(q-1)}] = 0.$$

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Note: Figure translations are in progress. See original paper for figures.