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Abstract

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ON THE THEORY OF BOUNDARY-VALUE PROBLEMS FOR GENERAL ELLIPTIC SYSTEMS

(Presented by Academician I. G. Petrovskii, 4 VII 1962)

In the present paper the theory of normal solvability of general boundary-value problems ([1–4]) is extended to elliptic systems of Douglis–Nirenberg ([5]) (and thereby also to systems of I. G. Petrovskii ([6])).

1. In a bounded domain $\mathfrak{D}$ of Euclidean space E_ν, of points $x = (x_1, \dots, x_\nu)$, a differential operator $A = A(x; D)$, $D = (D_1, \dots, D_\nu)$, $D_k = -i\,\partial/\partial x_k$, is given. Here $A(x; D)$ is a square matrix of order n; its elements $A_{ij}(x; D)$ are linear partial differential operators with smooth coefficients, of order $\alpha_{ij} \leq s_i + t_j$ ($s_1, \dots, s_n$, $t_1, \dots, t_n$ are integers, cf. ([7,8])). The principal part of A will be called the operator $\pi A(x; D)$ obtained after discarding in the operators A_{ij} all terms of order $< s_i + t_j$. We shall call the operator A elliptic if

$$\det \pi A(x; \xi) \neq 0 \quad \text{for } x \in \mathfrak{D}, \quad \xi = (\xi_1, \dots, \xi_\nu) \neq 0^*. \tag{1}$$

For $\nu > 2$, from the ellipticity condition it follows (as also in ([1])) that $\sum(s_i + t_i)$ (the order of the system) is an even number, i.e. $2r = \sum(s_i + t_i)$; for $\nu = 2$ this is additionally assumed.

On the boundary Γ of the domain $\mathfrak{D}$ a boundary operator $B = B(x; D)$ is given. Here $B(x; D)$ is a rectangular matrix of size $r \times n$, whose elements $B_{ij}(x; D)$ are linear differential operators of order β_{ij}. Let $m_i = \max(\beta_{ij} - t_j)$ over all $j = 1, \dots, n$. The principal part of B will be called the operator $\pi B(x; D)$, obtained by discarding in each element B_{ij} terms of order $< m_i + t_j^{**}$.

The operators A and B at each point Q of the boundary Γ are connected by the condition of Ya. B. Lopatinskii, which we formulate in the form indicated in ([4,9]). For simplicity of notation we shall assume that Q is at the origin of coordinates and that at Q the normal to Γ coincides with the x_ν-axis. Consider the operators with constant coefficients $A_0(D) = \pi A(0; D)$, $B_0(D) = \pi B(0; D)$. For an arbitrary real $\dot{\xi} = (\xi_1, \dots, \xi_{\nu-1})$, consider on the half-line $x_\nu \geq 0$ the boundary-value problem for a system of ordinary equations

$$A_0(\dot{\xi}, D_\nu)v(x_\nu) = 0, \quad x_\nu > 0; \tag{2}$$

$$B_0(\dot{\xi}, D_\nu)v(x_\nu) = h, \quad x_\nu = 0. \tag{3}$$

Here h is an r-dimensional numerical vector, and $\dot{\xi}$ plays the role of a parameter.

We shall say that condition L is satisfied at the point Q if problem (2), (3), for all h, is uniquely solvable in the class of functions tending to zero as $x_\nu \to +\infty$***.

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* Another (effective) form of this definition of ellipticity is indicated in ([7],[8]).

** We note that this method of isolating the highest-order terms in the boundary conditions may be used, proceeding from considerations expressed by I. M. Gel' fand in ([9]).

*** Denote by $\mathfrak{M}_+$ the space of solutions of system (2) tending to zero as $x_\nu \to +\infty$. For $\nu > 2$ one can prove that $\dim \mathfrak{M}_+ = r$; for $\nu = 2$ this must be assumed additionally.

2. The object of our study will be the boundary-value problem

$$A(x; D)u(x) = f(x), \qquad x \in \mathfrak{D}; \tag{4}$$

$$\lim_{x \to \dot{x}} B(x, D)u(x) = g(\dot{x}), \qquad \dot{x} \in \Gamma, \tag{5}$$

where $u(x) = \{u_1(x), \dots, u_n(x)\}$; $f(x) = \{f_1(x), \dots, f_n(x)\}$; $g(x) = \{g_1(x), \dots, g_r(x)\}$.

For the vector-functions u, f, g we introduce a special triple of functional spaces $\mathfrak{U}^{(l)}$, $\mathfrak{F}^{(l)}$, $\mathfrak{G}^{(l-1/2)}$. Let l be an integer, $l \geq l_0$, where

$$l_0 = \max\{-s_i, t_j, m_k + 1\} \qquad (1 \leq i, j, k \leq n).$$

Denote by $\mathfrak{U}^{(l)}$, $\mathfrak{F}^{(l)}$ the direct products of n Sobolev spaces $W_2^{(l)}(\mathfrak{D})$,

$$\mathfrak{U}^{(l)} = \prod_{j=1}^{n} W_2^{(l+t_j)}(\mathfrak{D}), \qquad \mathfrak{F}^{(l)} = \prod_{i=1}^{n} W_2^{(l-s_i)}(\mathfrak{D})$$

and define in them the norms*

$$\|u, \mathfrak{U}^{(l)}\| = \sum \|u_j, \mathfrak{D}\|_{l+t_j}; \qquad \|f, \mathfrak{F}^{(l)}\| = \sum \|f_i, \mathfrak{D}\|_{l-s_i}.$$

Denote by $\mathfrak{G}^{(l-1/2)}$ the direct product of r spaces of L. N. Slobodetskii [10], and define the norm in this space

$$\mathfrak{G}^{\,l-1/2} = \prod W_2^{(l-1/2-m_k)}(\Gamma), \qquad \|g, \mathfrak{G}^{(l-1/2)}\| = \sum \|g_k, \Gamma\|_{l-1/2-m_k}.$$

To the problem (4), (5) there corresponds a bounded operator $\mathfrak{A} = (A, B)$, acting from $\mathfrak{U}^{(l)}$ into the direct product $\mathfrak{F}^{(l)} \times \mathfrak{G}^{(l-1/2)}$.

The operator $\mathfrak{A}$ is called **elliptic** if the operator A is elliptic in the domain $\mathfrak{D}$, and at every point of the boundary Γ condition L is satisfied.

3. The following theorem is valid (see [4]).

Theorem. The following three assertions are equivalent:

a) the operator $\mathfrak{A} = (A, B)$ is elliptic,

b) if $u \in \mathfrak{U}^{(l_0)}$, $Au \in \mathfrak{F}^{(l)}$, $Bu \in \mathfrak{G}^{(l-1/2)}$, then $u \in \mathfrak{U}^{(l)}$,

$$\|u, \mathfrak{U}^{(l)}\| \le C\big[\|Au, \mathfrak{F}^{(l)}\| + \|Bu, \mathfrak{G}^{(l-1/2)}\| + \|u, \mathfrak{D}\|_0\big], \tag{6}$$

where the constant C does not depend on $u(x)$, and $\| \,, \mathfrak{D}\|_0$ is the usual norm in L_2;

c) the problem (4), (5) is normally solvable (i.e. the operator $\mathfrak{A}$ is a Φ-operator in the sense of [11])**.

As in simpler cases, in proving this theorem the center of gravity will lie in the study of the operator

$$\mathfrak{A}_0 = (A_0(D), B_0(D))$$

in the half-space $x_\nu \ge 0$. In this case we can make the Fourier transform and arrive at the problem (2), (3). M. S. Agranovich observed that for the fundamental matrix of the problem (2), (3) (in the case of systems (1)–(4)) one can construct an integral representation [4]***.

$$\Omega(x_\nu, \xi) = \int_\gamma e^{i\lambda x_\nu} A_0^{-1}(\xi, \lambda) N(\xi, \lambda)\, d\lambda, \tag{7}$$

* By $\| \,, \mathfrak{D}\|_k$ we denote the norm in the space $W_2^k(\mathfrak{D})$.

** $\mathfrak{A}$ is called a Φ-operator if the equation $\mathfrak{A}u = 0$ has in $\mathfrak{U}^{(l)}$ a finite number of linearly independent solutions, the range of the operator $\mathfrak{A}$ in $\mathfrak{F}^{(l)} \times \mathfrak{G}^{(l-1/2)}$ is closed, and the quotient space $\mathfrak{F}^{(l)} \times \mathfrak{G}^{(l-1/2)}/\mathfrak{A}\mathfrak{U}^{(l)}$ is finite-dimensional.

*** In carrying out this work the author used M. S. Agranovich' s report on the normal solvability of elliptic systems, delivered at the seminar of M. I. Vishik at Moscow State University in March 1962.

where γ is a Jordan contour in the half-plane $\operatorname{Im}\lambda > 0$, enclosing the zeros of the polynomial $\det A_0(\xi, \lambda)$, and $N(\xi, \lambda)$ is a rectangular matrix of size $n \times r$, whose entries are polynomials in λ and infinitely differentiable functions of ξ. The integral representation (7) makes it possible to transfer to systems (with the aid of simple modifications) the technique developed for a single equation ([4,12]).

4. We shall outline the construction of a matrix $N(\xi, \lambda)$ satisfying the conditions:

$$\int_\gamma B_0(\xi,\lambda)A_0^{-1}(\xi,\lambda)N(\xi,\lambda)\,d\lambda = E, \tag{8}$$

$$N(\alpha\xi, \alpha\lambda) = \alpha^{-1}S(\alpha)N(\xi,\lambda)M^{-1}(\alpha); \tag{9}$$

here E_r is the identity matrix of order r; $S(\alpha) = \|\delta_{ij}\alpha^{s_i}\|$, $M(\alpha) = \|\delta_{ij}\alpha^{m_i}\|$ are diagonal matrices of orders n and r.

The matrix $N(\xi, \lambda)$ must first of all be constructed for points of the unit sphere $|\xi| = 1$. Its continuation to the whole space is carried out by means of (9).

If $s_1 = \ldots = s_n$ (i.e., the system is elliptic in the sense of I. G. Petrovskii), then the matrix $N(\xi, \lambda)$ is constructed in the same way as in ([4]). Using condition Π, one can show that the rank of the matrix

$$\int_\gamma B_0(\xi,\lambda)A_0^{-1}(\xi,\lambda)(E, \lambda E, \ldots, \lambda^{R-1}E)\,d\lambda$$

is equal to r (here E is the identity matrix of order n, and R is the maximal order of differentiation in A). Let now ω be an arbitrary point on the unit sphere $|\xi| = 1$. For points ξ of this sphere close to ω, in the matrix $(E, \ldots, \lambda^{R-1}E)$ one can choose a minor $H_\omega(\lambda)$ of order r in such a way that the matrix

$$\Lambda_\omega(\xi) = \int_\gamma B_0(\xi,\lambda)A_0^{-1}(\xi,\lambda)H_\omega(\lambda)\,d\lambda$$

will be nonsingular. For these ξ we put $N_\omega(\xi, \lambda) = H_\omega(\lambda)\Lambda_\omega(\xi)$. Choosing on the sphere $|\xi| = 1$ a sufficiently fine partition of unity $1 = \sum \chi_\omega(\xi)$, we define $N(\xi, \lambda)$ by the formula

$$N(\xi,\lambda) = \sum \chi_\omega(\xi)N_\omega(\xi,\lambda).$$

5. We shall now show that, being able to construct the matrix $N(\xi, \lambda)$ for systems elliptic in the sense of Petrovskii, one can also construct it for the system (4), (5).

Let $P(\lambda) = \|P_{ij}(\lambda)\|$ be a polynomial (square) matrix, and for $i = 1, \ldots, n$ let the degree of $P_{ij} \le r_i$. Discarding in the elements $P_{ij}(\lambda)$ the terms of degree $< r_i$, we form the matrix $P'(\lambda)$. The matrix $P(\lambda)$ is called normal if $\det P'(1) \neq 0$. Observe that if the system A is elliptic in the sense of Petrovskii, then the matrix $A_0(\xi, \lambda)$ will be normal*.

Lemma. Let the matrix $A_0(\xi) = \|A_{ij}^{(0)}\|$ satisfy the conditions:

$$\text{degree } A_{ij}^{(0)} = s_i + t_j; \qquad \det A_0(\xi) \neq 0 \quad \text{for } \xi \neq 0.$$

* Here we regard the elements of the matrix $A_0(\xi, \lambda)$ as polynomials in λ; ξ plays the role of a "dimensionless" parameter.

There exists a (square) matrix $C(\dot{\xi}, \lambda)$ possessing the following properties:

1) the elements $C_{ij}(\dot{\xi}, \lambda)$ are polynomials in $\dot{\xi}, \lambda$ of degree $s_i - s$ (if $s_i < s_j$, then $C_{ij} \equiv 0$);
2) $\det C(\dot{\xi}, \lambda) = \text{const}$;
3) the matrix $C(\dot{\xi}, \lambda) A_0(\dot{\xi}, \lambda)$ (as a polynomial matrix in λ) is normal.

This lemma is proved with the aid of algorithm ([7]) (see Theorem 1). Put

$$\widetilde{A}_0(\dot{\xi}, \lambda) = S^{-1}(|\dot{\xi}|)\, C(\dot{\xi}, \lambda) A_0(\dot{\xi}, \lambda) S(|\dot{\xi}|), \qquad \widetilde{B}_0(\dot{\xi}, \lambda) = B_0(\dot{\xi}, \lambda) S(|\dot{\xi}|).$$

It follows from condition Π that the problem

$$\widetilde{A}_0(\dot{\xi}, D_\nu) v = 0, \qquad \widetilde{B}_0(\dot{\xi}, D_\nu) v = h$$

is uniquely solvable for all h in the class of functions tending to 0 as $x_\nu \to +\infty$. From item 4 follows the existence of a matrix $\widetilde{N}(\dot{\xi}, \lambda)$, where

$$\int_\gamma \widetilde{B}_0(\dot{\xi}, \lambda) \widetilde{A}_0^{-1}(\dot{\xi}, \lambda) \widetilde{N}(\dot{\xi}, \lambda)\, d\lambda = E.$$

The matrix

$$N(\dot{\xi}, \lambda) = C^{-1}(\dot{\xi}, \lambda) S(|\dot{\xi}|) \widetilde{N}(\dot{\xi}, \lambda)$$

will be the desired matrix satisfying (8), (9).

6. The a priori estimates (6) can also be constructed in the metrics L_p ($p \ge 1$). Applying to (7) the inverse Fourier transform, one can explicitly write out the matrix of the "Poisson kernels" and apply the technique ([12]).
7. As in ([4]), the results of this work can be extended to the case where the coefficients of the operator B are singular integral operators on Γ, and the theorems on the dependence of the index on the coefficients inside the domain ([14]) and on the dependence of the index on the boundary conditions (([4]), Theorem 2), considered in the present work for systems, can be carried over.

8. Using the technique ([13]), one can refine the a priori estimate (6) and prove analyticity of the solutions of problem (4), (5) (of course, under the assumption of analyticity of Γ and of all coefficients).

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REFERENCES

[1] Ya. B. Lopatinskii, *Ukr. Mat. Zh.*, **5**, 2 (1953).
[2] Z. Ya. Shapiro, *Izv. AN SSSR, Ser. Matem.*, **15**, 539 (1953).
[3] L. N. Slobodetskii, *Vestn. LGU, Ser. Matem.*, **7**, 2 (1960).
[4] M. S. Agranovich, A. S. Dynin, *DAN*, **146**, No. 3 (1962).
[5] A. Douglis, L. Nirenberg, *Comm. Pure and Appl. Math.*, **8**, 3 (1955).
[6] I. G. Petrovskii, *Matem. Sborn.*, **5**, 3 (1939).
[7] L. R. Volevich, *DAN*, **136**, No. 1 (1960).
[8] L. R. Volevich, *Matem. Sborn.*, **101**, 3 (1962).
[9] I. M. Gel' fand, *UMN*, **15**, 3 (1960).
[10] L. N. Slobodetskii, *Uch. Zap. Leningr. Gos. Ped. Inst. im. A. I. Herzen*, **197** (1958).
[11] M. G. Krein, I. Ts. Gokhberg, *UMN*, **12**, 2 (1957).
[12] S. Agmon, A. Douglis, L. Nirenberg, *Comm. Pure and Appl. Math.*, **12**, 4 (1959).
[13] C. B. Morrey, L. Nirenberg, *Comm. Pure and Appl. Math.*, **10**, 2 (1957).
[14] M. S. Agranovich, *DAN*, **142**, No. 5 (1962).

Note: Figure translations are in progress. See original paper for figures.