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Abstract

Full Text

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On a Problem of Neuwirth Concerning Knot Groups

(Presented by Academician P. S. Aleksandrov on 17 VII 1963)

In the book ⁽¹⁾, R. H. Fox proposed several problems on knots, among them the following Problem No. 3 (first posed by Neuwirth ⁽²⁾): Is it true that every knot group can be represented as the free product of two free groups of the same rank n with amalgamated subgroup of rank $2n - 1$?

Neuwirth ⁽²⁾ proved this assertion for all alternating knots and for knots with at most 9 crossings except 9_{46} (in the standard notation ⁽³⁾). Here we shall show that the assertion is also true for 9_{46} . Moreover, we shall strengthen it and prove that, in a certain sense, the converse assertion holds, namely, that every group having a representation of the strengthened form is the group of a knot in some three-dimensional manifold.

1. Let V be an oriented handlebody of genus n , and let k be a simple closed path on the surface. We shall say that k has **full rank** if its homotopy class in V cannot be represented by free generators fewer in number than n . If k does not cut the surface, then k can be completed to a canonical cut $k_1, k_2 = k^{-1}, k_3, \dots, k_{2n}$ of the surface (i.e. k_1, \dots, k_{2n} are simple closed paths which cut the surface into a disk with boundary

$$\prod_{i=1}^n b_{2i-1} b_{2i} b_{2i-1}^{-1} b_{2i}^{-1}.$$

By \mathfrak{F} we denote the fundamental group of the handlebody V with base point at the initial point of the path k . Let \mathbf{u}_k be the subgroup of the group \mathfrak{F} consisting of those classes which contain a path with the following properties: it lies on the surface, its intersection with k consists only of the initial point, where it only touches k . In addition, the path leaves k on one definite side. The following proposition holds:

In order that k have full rank, it is necessary and sufficient that \mathbf{u}_k have rank $2n - 1$.

Here we shall need only the sufficiency, which follows from the "loop theorem" of Papakyriakopoulos ⁽⁴⁾ and Dehn' s lemma.

Let M be a three-dimensional manifold and (V, W) a Heegaard splitting for M . Let a simple closed path k , not cutting F , be situated on the common

Fig. 1

Figure 1: Fig. 1

surface F of the handlebodies V and W . We obtain the group of the knot k in the following way: complete k to a canonical cut $k_1, k_2 = k^{-1}, \dots, k_{2n}$ of the surface and choose free generators S_1, \dots, S_n of the fundamental group of the handlebody V and T_1, \dots, T_n for W . Let K_1, \dots, K_{2n} be the words for the paths k_1, \dots, k_{2n} in the generators S_1, \dots, S_n , and let H_1, \dots, H_{2n} be the corresponding words in T_1, \dots, T_n . From van Kampen's theorem ⁽⁵⁾ it follows that the group \mathfrak{G} of the knot k has generators $S_1, \dots, S_n, T_1, \dots, T_n$ and defining relations $K_2 = H_2, \dots, K_{2n} = H_{2n}$. If k has full rank both in V and in W , then its group is the free product of two free groups of rank n with amalgamated subgroup of rank $2n - 1$.

More precisely, the following holds:

A. \mathfrak{G} contains two free subgroups \mathfrak{F}_1 and \mathfrak{F}_2 with free generators S_1, \dots, S_n and T_1, \dots, T_n . There exist in \mathfrak{F}_1 (and in \mathfrak{F}_2) elements K_1, \dots, K_{2n} (or H_1, \dots, H_{2n}) which generate \mathfrak{F}_1 (or \mathfrak{F}_2) and satisfy the equality

$$\prod_{i=1}^n [K_{2i-1}, K_{2i}] = 1 \quad \left(\text{or} \quad \prod_{i=1}^n [H_{2i-1}, H_{2i}] = 1 \right).$$

$K_2 = H_2, \dots, K_{2n} = H_{2n}$ are defining relations of the group \mathfrak{G} .

2. In the case of the knot 9_{46} we make the isotopic deformations shown in Fig. 1. In Fig. 1e we see that 9_{46} has the word $S_1 S_2^{-1} S_1^{-1} S_2$ on V and $T_1^{-1} T_2 T_1^{-1} T_2^{-1} T_1^2 T_2^{-1} T_1 T_2$ on W . Since both words have no period and both are not contained in a system of free generators, 9_{46} has full rank on both sides.

Fig. 1. $a-9_{46}$ in the standard projection ⁽³⁾; $b-d$ —deformed knots; $e-V$ and W denote full handlebodies, on their common surface lies the knot; the disks S_1 and S_2 form a cut for V ; T_1 and T_2 —a cut for W

In the ordinary projection (i.e. Fig. 1a) the natural Heegaard splitting for 9_{46} has genus 5; and therefore the relations obtained from it are not suitable for Neuwirth's assertion.

3. *If, conversely, the group \mathfrak{G} has form A, then it is the group of a knot lying on a Heegaard splitting of some manifold and having full rank on both sides.*

We take two full handlebodies V and W of genus n with surfaces F_1 and F_2 , and choose arbitrary systems of free generators S_1, \dots, S_n , or T_1, \dots, T_n , for the fundamental groups of the handlebodies V and W . Then there exist such canonical cuts k_1, \dots, k_{2n} and h_1, \dots, h_{2n} of the surfaces F_1 or F_2 that their paths induce, in the corresponding fundamental groups, the words K_1, \dots, K_{2n} or H_1, \dots, H_{2n} ⁽⁶⁾. We identify F_1 and F_2 by means of a homeomorphism that

maps k_i to h_i , and we obtain a three-dimensional manifold M . We regard k_2 as a knot in M . In this case \mathfrak{G} is its group.

Let k and k' be two knots whose groups have presentations of type A, obtained from canonical cuts of surfaces for Heegaard splittings. *If there exists an automorphism $\alpha : \mathfrak{G} \rightarrow \mathfrak{G}''$ with $\alpha K_i = K'_i$ and $\alpha H_1 = H_1^*$, then there exists a homeomorphism η between the manifolds containing k or k' , which carries k to k' .* η is constructed as follows. k and k' lie on the surfaces of Heegaard splittings (V, W) or (V', W') . First we construct

* By primes we denote objects for k' .

homeomorphism between the surfaces which carries the cut defining the presentation A for k into the cut corresponding to k' . This homeomorphism can be extended to homeomorphisms between V and V' and between W and W' ⁽⁶⁾.

From the presentation A for the knot group k one easily obtains the subgroup induced by the fundamental group of the surface of a regular neighborhood of k : it is generated by the elements $H_1^{-1}K_1$ and K_2 . Sometimes our conditions concerning the position of the knot on a Heegaard splitting surface are superfluous, as follows from Neuwirth's work ⁽⁷⁾. In this case, solely from knowledge of the generators and relations for the knot group, one can obtain that it lies on a Heegaard splitting surface having full rank on both sides.

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