S. A. LOMOV

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Abstract

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S. A. LOMOV

A POWER BOUNDARY LAYER IN PROBLEMS WITH A SMALL PARAMETER

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In this note we consider the construction of an asymptotic expansion in powers of the small parameter $\varepsilon > 0$ for the solution of the Cauchy problem and of a boundary-value problem for the equation

$$L_\varepsilon y \equiv (\varepsilon + x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + a_{n-2}(x)y^{(n-2)} + \dots + a_0(x)y = h(x). \quad (1)$$

Although for $\varepsilon = 0$ the order of equation (1) is not lowered, nevertheless one boundary condition is lost, since the equation

$$L_0 w \equiv xw^{(n)} + a_{n-1}(x)w^{(n-1)} + a_{n-2}(x)w^{(n-2)} + \dots + a_0(x)w = h(x) \quad (2)$$

is degenerating at the left endpoint of the interval $[0, a]$, on which we consider equation (1). Therefore in such problems (including equations in partial derivatives) the phenomenon of a boundary layer arises naturally, as in problems in which, for $\varepsilon = 0$, the order of the equation is lowered. Here the boundary layer has a power order of decrease as $\varepsilon \to 0$; more precisely, it has the form

$$P_m[\ln(1 + x/\varepsilon)]/(1 + x/\varepsilon)^c,$$

where $P_m(u)$ is a polynomial of degree m, $c > 0$, $P_m(0) \neq 0$. In considering the Cauchy problem we restrict ourselves to the case where $a_{n-1}(0) \equiv \alpha$ is not an integer. In what follows this must be kept in mind. The boundary-value problem is considered for $n = 2$, both for integral and nonintegral $a_1(0) > 1$.

1. Let us consider the behavior of the solution $y_\varepsilon(x)$ of equation (1) for sufficiently small ε under zero initial conditions:

$$y_\varepsilon(0) = y'_\varepsilon(0) = \dots = y_\varepsilon^{(n-1)}(0) = 0. \quad (1')$$

For fixed $\varepsilon > 0$, problem (1), (1′) has a unique solution. For $\varepsilon = 0$, equation (1) turns into equation (2), whose solution can no longer satisfy the conditions (1′), but does satisfy the conditions

$$w(0) = w'(0) = \dots = w^{(n-2)}(0) = 0, \qquad |w^{(n-1)}(x)| \le M. \tag{2'}$$

Theorem 1. *If the coefficients of equation (2)* $a_i(x), h(x) \in C^{(m+1)}(0,a)$*, and* $a_{n-1}(0) \equiv \alpha > 0$*, then equation (2) has a unique solution* $w_0(x) \in C^{(m+n)}(0,a)$ *satisfying the conditions:*

$$w_0^{(r)}(0) = D_r \quad (r = 0, 1, \dots, n-2); \qquad |w_0^{(n-1)}(x)| \le M. \tag{2”}$$

We take the solution $w_0(x)$ of the limiting problem (2), (2′) as a certain approximation to the solution $y_\varepsilon(x)$. Now it is necessary to eliminate the discrepancy in the last initial condition (1′). To this end we shall seek the solution of the problem (1), (1′) in the form $y_\varepsilon(x) = w_\varepsilon(x) + v_\varepsilon(x)$, so that $L_\varepsilon w_\varepsilon = h(x)$ and $L_\varepsilon v_\varepsilon = 0$. The point $x = -\varepsilon$ for the equation $L_\varepsilon v_\varepsilon = 0$ is a singular point. The roots of the indicial equation, which is introduced in the same way as in the analytic case, are

$$\rho_0 = 0,\ \rho_1 = 1, \dots, \rho_{n-2} = n-2,\ \rho_{n-1} = n - 1 - a_{n-1}(-\varepsilon).$$

The root ρ_{n-1} gives grounds to suppose that the boundary layer should be sought in the form

$$v_\varepsilon(x) = c\, v(x,\varepsilon)/(x+\varepsilon)^{a_{n-1}(-\varepsilon)-n+1},$$

$$(a_{n-1}(-\varepsilon) > 0,\ 0 \le \varepsilon < \varepsilon_0).$$

Subjecting $v_\varepsilon^{(n-1)}(x)$ to the condition

$$w^{(n-1)}(0) + v^{(n-1)}(0) = 0,$$

we find that

$$v_\varepsilon(x) = \varepsilon^{\,n-1} c_0 v(x,\varepsilon)/(1+t)^{a_{n-1}(-\varepsilon)-n+1} \qquad (t = x/\varepsilon).$$

Therefore in $L_\varepsilon v_\varepsilon(x)$ we substitute $v_\varepsilon(x) = v(x,\varepsilon)\,\bar v(t,\varepsilon)$. Taking into account that the function $\bar v(t,\varepsilon)$ satisfies any of the equations of the form

$$(1+t)\bar v^{(i)} + (a_{n-1}(-\varepsilon) - n + i)\bar v^{(i-1)} = 0, \quad i = 1, \dots, n,$$

and grouping the terms–

Taking into account their dependence on ε, we express the operator L_ε in terms of two operators M_ε and $\overline{L}_\varepsilon$:

$$L_\varepsilon v_\varepsilon(x) \equiv v M_\varepsilon v + \sum_{i=0}^{n-1} \varepsilon^{-n+i+1} K_i v \cdot (\overline{L}_\varepsilon \overline{v})^{(n-i-1)}, \tag{*}$$

where

$$M_\varepsilon v \equiv (\varepsilon + x) v^{(n)} + \sum_{k=0}^{n-1} b_{n-k-1}(x,\varepsilon) v^{(n-k-1)},$$

$$\overline{L}_\varepsilon \overline{v} \equiv (1+t)\overline{v}' + (a_{n-1}(-\varepsilon) - n + 1)\overline{v},$$

$$K_0 v \equiv v(x,\varepsilon), \qquad K_i v = C_n^{n-i} v^{(i)} + \sum_{k=0}^{i-1} d_{i-k-1}(x,\varepsilon) v^{(i-k-1)} \quad (i = 1, \dots, n-1),$$

$$b_{n-k-1}(x,\varepsilon) = A_{n-k-1}(x,\varepsilon)/(x+\varepsilon)^k, \qquad d_{i-k-1}(x,\varepsilon) = A^i_{n-k-1}(x,\varepsilon)/(x+\varepsilon)^{k+1}.$$

For brevity we do not write out the dependence of the sufficiently smooth functions $A_{n-k-1}(x,\varepsilon)$ and $A^i_{n-k-1}(x,\varepsilon)$ on the coefficients $a_i(x)$ and $a_{n-1}(-\varepsilon)$. We can now obtain separately equations for $v(x,\varepsilon)$ and $\overline{v}(t,\varepsilon)$, by putting $(\varepsilon + x)^{n-2} M_\varepsilon v = 0$ and $\overline{L}_\varepsilon \overline{v} = 0$. For $\varepsilon = 0$ these equations give zero approximations to the functions $v(x,\varepsilon)$ and $\overline{v}(t,\varepsilon)$:

$$\begin{aligned} M_0 v \equiv x^{n-1} v^{(n)} + x^{n-2} \overset{0}{\overline{b}}_{n-1}(x) v^{(n-1)} + \cdots \\ \cdots + x \overset{0}{\overline{b}}_2(x) v'' + \overset{0}{\overline{b}}_1(x) v' + \overset{0}{\overline{b}}_0(x) v = 0, \end{aligned} \tag{3}$$

$$\overline{L}_0 \overline{v} \equiv (1+t)\overline{v}' + (\alpha - n + 1)\overline{v} = 0, \tag{4}$$

where $\overset{0}{\overline{b}}_i(x) = x^{n-i-1} b_i^0(x)$, $i = 1, \dots, n-1$, $\overset{0}{\overline{b}}_0(x) = x^{n-2} b_0^0(x)$, and, for $\alpha > 0$, all $\overset{0}{\overline{b}}_i(0) \neq 0$, while $b^0_{n-k-1}(x) = b_{n-k-1}(x, 0)$. As the zero approximation to the function $v(x,\varepsilon)$ we take the solution $v_0(x)$ of equation (3) ($v_0(0) \neq 0$), continuous together with its derivatives up to a certain order; its existence will be established with the aid of the following facts. The indicial equation for equation (3) has the form:

$$\begin{aligned} \rho\big[(\rho-1)(\rho-2)\cdots(\rho-n+1) + (\rho-1)(\rho-2)\cdots(\rho-n+2)\overset{0}{\overline{b}}_{n-1}(0) + \cdots \\ \cdots + (\rho-1)\overset{0}{\overline{b}}_1(0) + \overset{0}{\overline{b}}_0(0)\big] = 0. \end{aligned} \tag{5}$$

Theorem 2. If the coefficients of the equation

$$x w^{(n)} + \sum_{i=0}^{n-1} a_i(x) w^{(i)} = 0, \tag{6}$$

$a_i(x) \in C^{(m+1)}_{(0,a)}$, $i = 0, 1, \dots, n-1$; $a_{n-1}(0) > 0$, then a fundamental system of solutions of equation (6) admits the representation: $w_i(x) = x^{\rho_i}\varphi_i(x)$, where $\rho_i = i$ for $i = 0, 1, \dots, n-2$; $\rho_{n-1} = n - 1 - a_{n-1}(0)$ are the roots of the indicial equation; $\varphi_i(x) \in C^{(m+n-i)}_{(0,a)}$; $\varphi_i(0) \neq 0$, if ρ_{n-1} is not equal to an integer.

Corollary. A fundamental system of solutions of equation (3), for noninteger $a_{n-1}(0)$, admits the representation $v_i(x) = x^{\overline{\rho}_i}\overline{\varphi}_i(x)$, where $\overline{\rho}_i$ are the roots of the indicial equation (5).

The function $v_0(x) = \overline{\varphi}_0(x)$ ($\overline{\rho}_0 = 0$) is the zero approximation to the function $v(x, \varepsilon)$.

A solution of equation (4) is the function $\overline{v}_0(t) = c_0/(1+t)^{\alpha-n+1}$. As a result, we obtain the zero approximation to the boundary-layer-type function $v_\varepsilon(x)$:

$$v_{\varepsilon 0}(x) = v_0(x)\overline{v}_0(t) \equiv v_{00}(x,t),$$

with whose aid we satisfy the last of the initial conditions (1′) by the term of lowest order in ε, i.e.

$$v_0(x)\left.\frac{d^{n-1}\overline{v}_0(t)}{dx^{n-1}}\right|_{\substack{t=0\\x=0}} = \frac{v_0(x)}{\varepsilon^{n-1}}\left.\frac{d^{n-1}\overline{v}_0(t)}{dt^{n-1}}\right|_{\substack{x=0\\t=0}} = -w^{(n-1)}(0).$$

Hence we find the constant $c_0 = \varepsilon^{n-1}\overline{c}_0$, where $\overline{c}_0$ is independent of ε. Thus, finally,

$$v_{\varepsilon 0}(x) = \varepsilon^{n-1}\overline{c}_0 v_0(x)/(1+x/\varepsilon)^{\alpha-n+1}$$

and the zero approxi-

···to the solution $y_\varepsilon(x)$ of problem (1), (1′) has the form $y_{\varepsilon 0}(x) = w_0(x) + v_{\varepsilon 0}(x)$. It can be shown, using the representation (*), that $L_\varepsilon y_{\varepsilon 0}(x) = h(x) + O(\varepsilon^{-n+2})$, while the residuals in the initial conditions have the form $y^{(k)}_{\varepsilon 0}(0) = O(\varepsilon^{n-k-1})$, $k = 0, 1, \dots, n-2$, $y^{(n-1)}_{\varepsilon 0}(0) = O(\varepsilon)$. To obtain the subsequent approximations we seek the functions

$$w_\varepsilon(x) = \sum_{i=0}^{n+m-1} \varepsilon^i w_i(x) + O(\varepsilon^{n+m}),$$

$$v(x,\varepsilon) = \sum_{i=0}^{n+m-1} \varepsilon^i v_i(x) + O(\varepsilon^{n+m}), \qquad \bar{v}(t,\varepsilon) = \sum_{i=0}^{n+m-1} \varepsilon^i \bar{v}_i(t) + O(\varepsilon^{n+m})$$

and in the usual way obtain equations for determining $w_k(x)$, $v_k(x)$, $\bar{v}_k(t)$:

$$L_0 w_k(x) = -w_{k-1}^{(n)}(x), \qquad M_0 v_k = -\sum_{j=0}^{n}\sum_{i=1}^{k} b_j^i(x) v_{k-i}^{(j)}, \qquad \bar{L}_0 \bar{v}_k(t) = -\sum_{i=1}^{k} a_{n-1}^i \bar{v}_{k-i}(t),$$

$$k = 1, \dots, n+m-1,$$

where $b_j^i(x)$ and a_{n-1}^i are the coefficients of the expansions of the corresponding coefficients in the operators $(\varepsilon + x)^{n-2} M_\varepsilon$ and $\bar{L}_\varepsilon$.

We find the functions $w_l(x)$, $\bar{v}_l(t)$, and $v_l(x)$ respectively under the initial conditions*:

$$1) \quad w_l^{(r)}(0) = 0, \quad r = 0, \dots, n-l-2, \quad l = 0, \dots, n-2;$$

$$w_l^{(r)}(0) = -\sum_{k=0}^{l-1} C_r^{n+k-l-1} \sum_{i=0}^{k} v_{k-i}^{(l+r-n-k+1)}(0) V_i^{(n+k-l-1)}(0),$$

$$n-l-1 \le r \le n-2, \quad 1 \le l \le n+m-1; \qquad |w_l^{(n-1)}(x)| \le M_l, \quad l = 0, 1, \dots, n+m-1;$$

$$2) \quad w_l^{(n-1)}(0) + \sum_{k=0}^{l} C_{n-1}^{n+k-l-1} \sum_{i=0}^{k} v_{k-i}^{(l-k)}(0) V_i^{(n+k-l-1)}(0) = 0, \quad 0 \le l \le n+m-1;$$

$$3) \quad v_l(0) = 1,$$

and the $v_l(x)$ have the necessary number of continuous derivatives.

Here

$$V_i(t) = \sum_{k=0}^{i} c_k \ln^k(1+t)/(1+t)^{\alpha-n+1},$$

i.e., $\bar{v}_i(t) = \varepsilon^{n-1} V_i(t)$ (the c_k do not depend on ε). As a result we can formulate the following theorem.

Theorem 3. If in equation (1)

$$a_{n-1}(x) \in C^{((m+n)(n-1)+1+n)}(0, a),$$

$$a_i(x) \in C^{((m+n)(n-1)+n)}(0,a), \quad i = 0, 1, \dots, n-2, \quad a_{n-1}(0) > 0, \quad h(x) \in C^{(n+m+1)}(0,a),$$

then the solution of problem (1), (1′) for sufficiently small $\varepsilon > 0$ can be represented in the form

$$y_\varepsilon(x) = \sum_{k=0}^{n+m-1} \varepsilon^k \left[w_k(x) + v_{0k}(x;t)\right] + z_\varepsilon(x) = y_{\varepsilon,n+m-1}(x) + z_\varepsilon(x),$$

where

$$L_\varepsilon y_{\varepsilon,n+m-1}(x) = h(x) + O(\varepsilon^{m+1}), \qquad y^{(i)}_{\varepsilon,n+m-1}(0) = O(\varepsilon^{n+m}), \quad i = 0, 1, \dots, n-1,$$

where $w_k(x) \in C^{(2n+m-k-1)}(0,a)$; the derivatives of order $n-1$ of

$$v_{0k}(x;t) = \sum_{i=0}^{k} v_{k-i}(x)\bar{v}_i(t)$$

are boundary-layer-type functions of power order in a neighborhood of the point $x = 0$ and uniformly tend to zero as $\varepsilon \to 0$ on any interval $[a_1, a]$ $(0 < a_1 < a)$; the remainder term $z_\varepsilon(x)$ and all its derivatives up to order $n-1$, as $\varepsilon \to 0$, are $O(\varepsilon^{m+1})$.

2. Consider the following boundary-value problem:

$$L_\varepsilon y \equiv (\varepsilon + x)y'' + a_1(x)y' + a_0(x)y = h(x), \tag{7}$$

$$y_\varepsilon(0) = y_0, \qquad y_\varepsilon(1) = y_1, \tag{7'}$$

where $a_1(0) > 1$, $a_0(x) \le 0$. Denote $a_1(x) = 1 + k(x)$, $k_0 = k(0) > 0$. For $\varepsilon = 0$, equation (7) takes the form

$$L_0 w \equiv xw'' + (1 + k(x))w' + a_0(x)w = h(x). \tag{8}$$

Problem (8), (7′) has no solution, but for equation (8) there is a unique solution to the problem in the following formulation:

$$|w_0(x)| \le M, \quad 0 \le x \le 1; \qquad w_0(x_0) = y_1, \quad 0 \le x_0 \le 1. \tag{8'}$$

[*] Terms with negative upper indices are replaced by zeros.

Using the representation (*) for $n = 2$, we obtain the equation for $v(x, \varepsilon)$ and $\bar{v}(t, \varepsilon)$:

$$M_\varepsilon v \equiv (\varepsilon+x)v''+(1+k(x)-2k(-\varepsilon))v'+\left(a_0(x)+\frac{k(-\varepsilon)(k(-\varepsilon)-k(x))}{x+\varepsilon}\right)v=0; \tag{9}$$

$$\bar{L}_\varepsilon \bar{v} \equiv (1+t)\bar{v}' + k(-\varepsilon)\bar{v} = 0.$$

1) In the case of nonintegral k_0, as in the Cauchy problem, we find $v_0(x)$ and $\bar{v}_0(t) = c_0/(1+t)^{k_0}$ $(v_0(0) = 1,\ \bar{v}_0(0) = y_0 - w_0(0))$ and set $y_{\varepsilon 0}(x) = w_0(x) + v_0(x)\bar{v}_0(t)$.

Having found $w_1(x)$, $v_1(x)$, and $\bar{v}_1(t) = (c_1+c\ln(1+t))/(1+t)^{k_0}$, and subjecting $\bar{v}_1(t)$ to the condition $v_1(0) = -w_1(0) - \bar{v}_0(0)$, we observe that discrepancies of a new kind appear: $y_{\varepsilon 1}(1) = y_1 + \varphi_0(\varepsilon)\varepsilon^{k_0} + \varphi_1(\varepsilon)\varepsilon^{k_0+1}\ln\varepsilon$; therefore it is necessary to introduce into the expansions of the functions $w_\varepsilon(x)$ and $v_\varepsilon(x)$ terms with $\varepsilon^{ik_0+j}\ln^k\varepsilon$, and to expand the functions $\varphi_i(\varepsilon)$ in powers of ε.

2) In the case of integral k_0, the first iterative process is carried out as in the nonintegral case, since the proof of Theorem 1 for $n = 2$ remains valid also for integral $k_0 > 0$; only the expansion is conducted in powers of $\varepsilon\ln^i\varepsilon$. It can be proved that equation (9) has two linearly independent solutions of the form $v_1(x, \varepsilon) = (x+\varepsilon)^{k(-\varepsilon)}w(x, \varepsilon)$ and

$$v_2(x,\varepsilon) = z(x,\varepsilon) + v_1(x,\varepsilon)\int_{x_0}^{x}\frac{dt}{(t+\varepsilon)^{k(-\varepsilon)-k_0+1}},$$

where $w(-\varepsilon, \varepsilon) \neq 0,\ z(-\varepsilon, \varepsilon) \neq 0\ (0 \le \varepsilon < \varepsilon_0)$. For $z(x, \varepsilon)$ there arises a differential equation of second order, and $z(x, \varepsilon)$ is sought in the form of a polynomial in powers of $\varepsilon\ln^i\varepsilon$. Up to powers ε^{k_0-1}, we shall take as approximations of a boundary-layer function

$$v_\varepsilon(x) = z(x,\varepsilon)\bar{v}(t,\varepsilon) + w(x,\varepsilon)\int_0^t\frac{(1+u)^{k_0-1}\,du}{(1+u)^{k(-\varepsilon)}},$$

the corresponding approximations for the product $z(x, \varepsilon)\cdot\bar{v}(t, \varepsilon)$. With the appearance in this product of the power ε^{k_0-1}, it is also necessary to take the integral term into account. Thus, in cases 1) and 2) we arrive at the theorem:

Theorem 4. If the coefficients of equation (7) satisfy $k(0) > 0$, $a_0(x) \le 0$, and

a) for nonintegral $k_0 = k(0)$: $k(x) \in C^{(m+2)}(0,1)$; $a_0(x), h(x) \in C^{(m+1)}(0,1)$;

b) for integral k_0: $k(x) \in C^{(2m+3)}(0,1)$; $a_0(x), h(x) \in C^{(2m+2)}(0,1)$, then the solution of equation (7) under conditions (7′), for sufficiently small ε, is representable in the form

$$y_\varepsilon(x) = \sum_{i,j,k=0}^{l,m,p} \varepsilon^{ik_0+j} \ln^k \varepsilon \, [w_{ij}^k(x) + u_{ij}^k(x;t)] + z_\varepsilon(x)$$

in the case of nonintegral k_0, and in the form

$$y_\varepsilon(x) = \sum_{i,j=0}^{m,l} \varepsilon^i \ln^j \varepsilon \, [w_{ij}(x) + u_{ij}(x;t)] + z_\varepsilon(x)$$

in the case of integral k_0, where the functions $w_{ij}^k(x) \in C^{(m-j+2)}(0,1)$, $w_{ij}(x) \in C^{(m-i+2)}(0,1)$; $u_{ij}^k(x;t)$ and $u_{ij}(x;t)$ are boundary-layer functions of power type in a neighborhood of the point $x = 0$, and the remainder term $z_\varepsilon(x)$, together with its derivative, has the following order in ε: $z_\varepsilon(x) = O(\varepsilon^m)$, $z'_\varepsilon(x) = O(\varepsilon^{m-1})$.

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Moscow Power Engineering Institute

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Note: Figure translations are in progress. See original paper for figures.