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Abstract

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MATHEMATICS

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ON THE DETERMINATION OF THE CLASS OF CONVERGENCE OF AN INTERPOLATION SERIES FOR THE ABEL-GONCHAROV AND GELFOND PROBLEMS

(Presented by Academician I. M. Vinogradov on 29 IX 1962)

Consider a system of linear functionals $\{L_n(F)\}$, defined on the entire class of entire functions $F(z)$ as follows:

$$L_n(F) = \frac{1}{2\pi i} \int_C F(z)\, u_n\left(\frac{1}{z}\right) \frac{dz}{z},$$

where

$$u_n(z) = z^n + \sum_{k=1}^{\infty} a_{n,k} z^{n+k}.$$

We shall suppose that the functions $u_n(z/\lambda_n)$, for $n > n_0(z)$, have in a certain domain G no zeros different from $z = 0$, and, moreover, uniformly with respect to z inside G,

$$\frac{u_{n+1}(z/\lambda_{n+1})}{u_n(z/\lambda_n)} \to u(z), \tag{1}$$

with $u(0) = 0,\ u'(0) = 1$.

On the sequence λ_n we impose the following restrictions:

1. $\lambda_n = \lambda(n)$, where $\lambda(z)$ is regular in the whole plane with the cut $(-\infty, 0)$, and $\lambda(z) = w = re^{i\theta}$ maps the half-plane $\operatorname{Re} z > 0$ onto a certain domain Ω, which is bounded by curves having, in a neighborhood of $w = \infty$, the equations $\theta = \pm\frac{1}{2}\alpha(r)$; $\lambda(z) > 0$ for $z > 0$.

2. $\alpha(r),\ \alpha'(r),\ \alpha''(r)$ are monotone for $r > r_1$.

3. There exists

$$\lim_{r\to\infty} \frac{\pi}{\alpha(r)} = \rho, \qquad \frac{1}{2} < \rho < \infty.$$

4. $\alpha(r)$ is a slowly increasing function, i.e.

$$\lim_{r\to\infty} r\,\frac{\alpha'(r)}{\alpha(r)} = 0.$$

Consider the entire function $\Phi(z)$, defined by the equality

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{e^{h(t)}}{t-z}\,dt, \tag{2}$$

where $h(z)$ is that branch of the function inverse to $\lambda(z)$ which takes positive values for $z > 0$. The contour L begins at infinity below the real axis, coinciding in a neighborhood of $t = \infty$ with the curve $\theta = -\frac{1}{2}\alpha(r)$, and ends at infinity above the real axis,

coinciding in a neighborhood of $t = \infty$ with the curve $\theta = -\frac{1}{2}\alpha(r)$. In the finite part of the plane the contour is drawn so that the point $t = z$ remains to the left of it.

With the aid of the function $\Phi(z)$, the desired class of entire functions $F(z)$ is singled out for which the interpolation series converges

$$F(z) = \sum_{n=0}^{\infty} L_n(F)P_n(z), \tag{3}$$

where $\{P_n(z)\}$ is a system of polynomials biorthogonal to the system of functionals, i.e. $L_n(P_m) = \delta_{n,m};\ n, m = 0, 1, 2, \ldots.$

Let the entire function $F(z)$ be representable in the form

$$F(z) = \frac{1}{2\pi i}\int_C \Phi(z\zeta)f(\zeta)\,d\zeta, \tag{4}$$

where $f(\zeta)$ is regular outside some domain D, i.e. $f(z)$ is Φ-associated with the entire function $F(z)$. To prove the convergence of the series (3), it is enough to show that the system of functions

$$v_n(\zeta) = \frac{1}{2\pi i}\int_C u_n\left(\frac{\zeta}{t}\right)\Phi(t)\,\frac{dt}{t} \tag{5}$$

forms a basis in the domain D, since in this case $\Phi(z\zeta)$ can be expanded in a uniformly convergent series:

$$\Phi(z\zeta) = \sum_{n=0}^{\infty} P_n(z) v_n(\zeta).$$

The results of the papers ([4,5]) make it possible to assert that the system (5) forms a basis in the domain D, if the function $v(z)$, defined by the equalities

$$v(z) = u[\omega(z)] e^{\frac{1}{\rho}\varphi[\omega(z)]}, \qquad \varphi(t) = t\frac{u'(t)}{u(t)},$$

$\omega(z)$ being the solution of the equation

$$t[\varphi(t)]^{1/\rho} = \rho^{1/\rho} z, \qquad \omega(0) = 0,$$

is regular and univalent in the star-shaped domain $D \subset G$, mapped by the function $w = v(z)$ onto a disk. Moreover, in the domain D the condition

$$\operatorname{Re}\left[z\frac{v'(z)}{v(z)}\right] > 0 \tag{6}$$

must be satisfied.

We now turn to the Abel-Goncharov problem. For it

$$L_n(F) = \frac{1}{2\pi i}\int_C \frac{F(z)}{(z-\lambda_n)^{n+1}}\, dz, \qquad u_n(z) = \frac{(-1)^{n+1}}{\lambda_n^{n+1}}\omega_n(z),$$

where

$$\omega_n(z) = \frac{z^n}{(z - 1/\lambda_n)^{n+1}},$$

and, by virtue of the conditions on λ_n, for $\omega_n(z)$ (1) holds, i.e.

$$\frac{\omega_{n+1}(z/\lambda_{n+1})}{\omega_n(z/\lambda_n)} \longrightarrow \frac{z}{z-1}.$$

Consequently, for the Abel-Goncharov problem

$$u(z) = \frac{z}{z-1}, \qquad \varphi(z) = \frac{1}{1-z},$$

$$v(z) = [1 - \tau(z)]e^{\frac{1}{\rho}\tau(z)}, \qquad \tau(z) = \frac{1}{1 - \omega(z)}, \tag{7}$$

where $\omega(z)$ is the solution of the equation

$$\frac{t^\rho}{1-t} = \rho z^\rho, \qquad \omega(0) = 0,$$

whence

$$z = \rho^{-1/\rho}\omega(z)[1 - \omega(z)]^{-1/\rho}. \tag{8}$$

To determine the domain D, let us expand z in a series in powers of $v(z)$ and determine the radius of convergence of this series. Setting $z = \sum_{n=0}^{\infty} c_n v^n$ and using (7) and (8), we find that

$$c_n = e^{-n/\rho}\rho^{-1/\rho}\frac{1}{2\pi i}\int \frac{(t/\rho - 1)(1-t)^{1/\rho - 1}e^{\frac{t}{\rho}n}}{t^n}\,dt.$$

Let us consider separately the cases $\rho < 1$, $\rho > 1$, and $\rho = 1$.
For $\rho < 1$, by the saddle-point method one can obtain the asymptotic estimate

$$|c_n| \sim \frac{\rho^{2-1/\rho}(1-\rho)^{1/\rho - 3}}{\sqrt{2\pi}}\, n^{-5/2}\left(\frac{e^{1-1/\rho}}{\rho}\right)^n,$$

whence

$$\overline{\lim_{n\to\infty}} |c_n|^{1/n} = \frac{e^{1-1/\rho}}{\rho}.$$

Consequently, for $\rho < 1$, $z(v)$ is regular in the disk

$$|v| < \rho e^{1/\rho - 1}, \tag{9}$$

and the inverse function, i.e. $v(z)$, is univalent in the domain obtained by mapping the disk (9) by means of the function $z(v)$.

For $\rho > 1$, c_n can be represented in the form

$$c_n = \frac{\rho^{-1/\rho}e^{-n/\rho}}{\Gamma(1-1/\rho)}\left(\frac{n}{\rho}\right)^{n-1/\rho}\frac{1}{(n-1)!}\int_0^\infty x^{-1/\rho}(1+x)^{-1}\left[\frac{n-1}{n(1+x)} - 1\right]e^{-\frac{n}{\rho}x + n\ln(1+x)}\,dx.$$

Estimating the integral by Laplace' s method, we obtain

$$|c_n| \sim \frac{\rho^{-1}}{\Gamma(1-1/\rho)}(\rho-1)^{1-1/\rho}n^{-1/\rho},$$

whence

$$\overline{\lim_{n\to\infty}}\,|c_n|^{1/n} = 1.$$

Thus, for $\rho > 1$, $v(z)$ is univalent in the domain obtained by mapping the disk $|v| < 1$ by means of the function $z(v)$.

The case $\rho = 1$ is especially simple; in this case $v(z) = -ze^{1+z}$, and $v(z)$ is univalent in the domain $|v(z)| < 1$. Condition (6), as is not hard to verify, is satisfied in the domain D in each of the cases under consideration and guarantees the starlikeness of D with respect to the point $z = 0$.

Theorem 1. Let $v(z)$ be the function defined by conditions (7), (8), and let D be the domain consisting of points that satisfy the inequality $|v(z)| < \delta_\rho$ ($\delta_\rho = \rho e^{1/\rho-1}$ for $\rho \leqslant 1$; $\delta_\rho = 1$ for $\rho \geqslant 1$) and form a connected set containing the point $z = 0$.

If the entire function $F(z)$ can be represented in the form (4), where $\Phi(z)$ is defined by formula (2), $f(\xi)$ is regular outside D, and the contour C contains all the singular points of $f(\xi)$, then the Abel–Goncharov series

$$\sum_{n=0}^{\infty} L_n(F)P_n(z)$$

with interpolation nodes λ_n converges to $F(z)$; moreover, the latter series converges not only in every finite part of the plane, but also in the sense of the topology $U(\Phi, D)$ [(3)]. (In this respect the result obtained is stronger than the results set forth in (1)–(3).)

A generalization of the Abel–Goncharov problem is the Gelfond interpolation problem [(6)], for which

$$L_n(F) = \frac{1}{2\pi i}\int_C \frac{F(z)}{\prod_{k=0}^{n}(z-\lambda_{n,k})}\,dz.$$

We shall require that the following conditions be satisfied:

$$\lambda_{n,k} - \lambda_n = \lambda_n \alpha_{n,k}, \qquad \alpha_{n,k} \to 0, \tag{10}$$

$$\sum_{k=0}^{n} |\alpha_{n+1,k} - \alpha_{n,k}| \to 0 \quad \text{as } n \to \infty, \tag{11}$$

and that λ_n satisfy the same conditions as in the Abel–Goncharov problem. Condition (10) means that the interpolation nodes $\lambda_{n,k}$ lie in the disk with center λ_n, whose radius is $o(\lambda_n)$, while condition (11) is a smoothness condition on the quantities $\alpha_{n,k}$.

Theorem 2. If the interpolation nodes in the Gelfond problem satisfy conditions (10), (11), where the restrictions indicated above are imposed on λ_n, then, for the Gelfond interpolation series, Theorem 1 holds.

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Note: Figure translations are in progress. See original paper for figures.