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SPECTRAL ANALYSIS OF BOUNDED NON-SELF-ADJOINT OPERATORS IN A SPACE WITH AN INDEFINITE METRIC

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In this note a characteristic matrix-function is constructed and the spectrum of a broad class of bounded non-self-adjoint operators acting in a space with an indefinite metric Π is studied. In doing so we do not additionally assume that the space Π contains only finitely many negative (or positive) lineals. We note that in a Hilbert space results analogous to those given here were obtained in the works of M. S. Livshits ([1,2]) and M. S. Brodskii ([2–4]).

The method that we shall use (the method of characteristic matrix-functions) was first applied in a space with an indefinite metric in the author' s works ([5–7]) in the study of nonunitary operators. In those works, however, it was assumed that the space Π satisfies the axioms of I. S. Iokhvidov and M. G. Krein ([8]) (i.e., is a space with an indefinite metric "of finite rank"). Here this is not assumed.

1. Let H be a Hilbert space, (f, g) the scalar product in H, and P a bounded invertible Hermitian operator acting in H. We introduce a metric in the space H by setting $[f, g] = (Pf, g)$. The scalar product $[f, g]$ has the same properties as (f, g), with one exception: the scalar square $[f, f]$ may be negative or equal to zero for $f \neq 0$. In what follows, the space H with scalar product $[f, g]$ will be denoted by Π, and we shall speak correspondingly of the H-metric and the Π-metric. Further, the notation $\Pi_2 = \Pi[-]\Pi_1$ means that the manifold Π_2 is orthogonal in the Π-metric to the manifold Π_1 ($\Pi_k \subset \Pi$). The notation $\Pi_1[+]\Pi_2$ means that the manifolds Π_1 and Π_2 are orthogonal in the Π-metric and linearly independent. In what follows we use the following assertion: if Π_1 is a nondegenerate subspace of the space Π (i.e., from $\varphi \in \Pi_1$ and $[\varphi, \Pi_1] = 0$ it follows that $\varphi = 0$), then $\Pi_2 = \Pi[-]\Pi_1$ is also a nondegenerate subspace, and moreover $\Pi = \Pi_1[+]\Pi_2$.
2. Let A be a linear bounded operator acting in Π (boundedness, continuity, and closedness of an operator are understood in the H-metric). The operator adjoint to A in the Π-metric (in the H-metric) will be denoted by A^0

(A^*). Then $A^0 = P^{-1}A^*P$. In this case the properties of the adjunction signs 0 and $*$ are analogous.

Let $E_A = \Pi[-]G_A$, where G_A is the totality of vectors f in Π such that $Af = A^0 f$. The operator A is called a K^r-operator if $\dim E_A = r$. In this case the operator

$$\frac{A - A^0}{i}$$

can be represented in the form

$$\frac{A - A^0}{i} \sum_{k,i=1}^{s} [\cdot, g_k] J_{ki} g_i,$$

where $s \geq r$, and $J = \|J_{ki}\|$ is a certain Hermitian and unitary matrix (the coefficient matrix). The collection of vectors $\{g_k\}_{k=1}^{s}$ is called an α-basis of the operator A. The α-basis of the operator A may be chosen so that its linear span coincides with E_A (as will also be assumed in what follows).

Definition. The matrix-function $\chi_A(\lambda)$, defined by the relation

$$\chi_A(\lambda) = I + i \left\| [(A^0 - \lambda I)^{-1} g_k, g_i] \right\| J,$$

where $\{g_k\}_1^s$ is an α-basis of the operator A, and J is the corresponding coefficient matrix, is called the **characteristic matrix-function** of the operator A.

The characteristic matrix-function of the operator A is analytic in the set of regular points of the operator A^0 and satisfies the relation $\chi_A(\lambda) J \chi_A^*(\bar{\lambda}) = J$. Moreover $\chi_{A^0}(\lambda) = \chi_A^{-1}(\lambda)$.

Let us agree to call an eigenvalue λ of the operator A a $(+)$-eigenvalue if the corresponding eigenvector f is positive (i.e. $[f, f] > 0$). The notions of $(-)$-eigenvalues and (0)-eigenvalues are introduced analogously.

Theorem 1. *Suppose that the nonreal spectrum K^r of the operator A consists only of eigenvalues of this operator.*

Then every nonreal zero of the function $\det \chi_A(\lambda)$ *is a $(+)$-eigenvalue or a $(-)$-eigenvalue of the operator A. Conversely, if λ_0 is a nonreal $(+)$-eigenvalue or $(-)$-eigenvalue of the operator A and, moreover, $\bar{\lambda}_0$ is a regular point of the operator A^0, then* $\det \chi_A(\lambda_0) = 0$.

We note that in the case when $r = 1$, the condition imposed on $\bar{\lambda}_0$ may be omitted.

3. Let $\Pi_p = \Pi[-]\Pi_A$, where Π_A is the largest invariant subspace of G_A in which the operator A induces a self-adjoint (in the Π-metric) operator. The subspace Π_p coincides with the closed linear span of $\{A^n g_k\}$, where $k = 1, 2, \dots, s$; $n = 0, 1, 2, \dots$. The operator $A_p = A|_{\Pi_p}$ is called the simple part of the operator A. Further, let the operator V map Π_1 onto Π_2 and, for any f and g from Π_1, satisfy the condition

$$[Vf, Vg]_2 = \theta[f, g]_1 \qquad (\theta = \pm 1).$$

Then for $\theta = 1$ the operator V is called **isometric**, and for $\theta = -1$, **coisometric**. Operators A_1 and A_2, acting respectively in the spaces Π_1 and Π_2, are called **isomorphic** if there exists an isometric operator V mapping Π_1 onto Π_2 (or Π_2 onto Π_1) such that $VA_1 = A_2V$ (or $A_1V = VA_2$). Coisomorphism of operators is defined analogously.

Theorem 2. *Suppose $\chi_{A_1}(\lambda) \equiv \chi_{A_2}(\lambda)$. Then, if $J_2 = J_1$, the simple parts of the operators A_1 and A_2 are isomorphic, while if $J_2 = -J_1$, they are coisomorphic.*

We note that in a space with an indefinite metric, for the characteristic matrix-function of a non-self-adjoint operator the identity $\chi_A(\lambda) \equiv I$ is possible.* It turns out that in this case the subspace Π_p (on which the simple part of the operator A is defined) coincides with its isotropic part and, consequently, is a nontrivial invariant subspace of the operator A.

4. Consider in the subspace Π_1 of the space Π the operator $A_1 = P_1 A$, where P_1 is the projection operator in Π onto Π_1. The α-basis of the operator A_1 consists of the vectors $g_{1k} = P_1 g_k$ $(k = 1, 2, \dots, s)$, where $\{g_k\}_1^s$ is the α-basis of the operator A. In this case the coefficient matrices J_1 and J, corresponding to the indicated α-bases, coincide. By analogy with how this was done in (4), the characte-

* I. S. Iokhvidov drew my attention to this possibility.

the characteristic matrix-function $\chi_{A_1}(\lambda) = I + i\left\|[(A^0 - \lambda I)^{-1} g_{1k}, g_{1i}]\right\| J_1$ of the operator A_1 will be called the **projection** of the matrix-function $\chi_A(\lambda)$ onto the subspace Π_1.

Theorem 3 (multiplication theorem). *Let Π_1 be a nondegenerate subspace of the operator A and $\Pi_2 = \Pi \ominus \Pi_1$. Then an arbitrary characteristic matrix-function of the operator A is the product of its projections onto Π_1 and Π_2.*

Using this theorem and the preceding results, we arrive at the relation

$$\sum_{k=1}^{m} |\operatorname{Im} \lambda_k| \leqslant \|P\| \sum_{k=1}^{s} \|g_k\|^2, \tag{1}$$

where $\{\lambda_k\}_{k=1}^m$ is the set of (+)-eigenvalues and (−)-eigenvalues of the operator A, and $\|P\|$ and $\|g_k\|$ are the norms of the operator P and the vector g_k in the H-metric.

It follows from relation (1) that the limit points of the set of (+)-eigenvalues and (−)-eigenvalues of the operator A can lie only on the real axis.

5. Let us now consider the case when the coefficient matrix J is Hermitian positive (or Hermitian negative). Without loss of generality, we may assume that $J = \theta I$, where $\theta = \pm 1$. In this case:

I. The (+)-eigenvalues of the K^r-operator A lie in the domain $\theta \operatorname{Im} \lambda \geqslant 0$, and the (−)-eigenvalues lie in the domain $\theta \operatorname{Im} \lambda \leqslant 0$.

II. The eigenvectors of the K^r-operator A corresponding to (0)-eigenvalues or to real eigenvalues belong to the subspace G_A.

III. A simple K^r-operator A has no (0)-eigenvalues, nor any real eigenvalues.

6. The following necessary condition for completeness holds:

Theorem 4. *Let the system of nonzero root vectors of the K^r-operator A be complete in the space* Π. *Then*

$$2\sum_{k=1}^{N} \operatorname{Im} \lambda_k = \sum_{k=1}^{s} [g_k, g_k] J_k \quad (N \leqslant \infty),$$

where $\{\lambda_k\}_{k=1}^N$ is the set of all (+)-eigenvalues and (−)-eigenvalues of the operator A, and $\{g_k\}_{k=1}^s$ is an α-basis of this operator.

7. In conclusion, we point out one problem important for the questions under consideration. There is reason to suppose that an arbitrary characteristic matrix-function of a K^r-operator A can be represented as the product of a J-nonexpanding and a J-noncompressing matrix-function. In other words, there is reason to suppose that, for the matrix-functions under consideration, there holds a matrix analogue of the well-known theorem of R. Nevanlinna stating that an arbitrary function of bounded type can be represented as the quotient of two functions bounded in some domain. The solution of this problem will make it possible to obtain a spectral decomposition (triangular model) of K^r-operators in a space with indefinite metric, and also to clarify the question of the behavior of the spectrum of an arbitrary bounded K^r-operator.

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Note: Figure translations are in progress. See original paper for figures.