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Abstract

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ON THE DIRICHLET PROBLEM FOR ELLIPTIC EQUATIONS OF ARBITRARY ORDER IN UNBOUNDED DOMAINS

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In the present paper a necessary and sufficient condition is found for the existence of a generalized solution of the homogeneous Dirichlet problem for the polyharmonic equation $\Delta^l u = f$ in an n-dimensional domain Ω for all $f \in L_q(\Omega)$ $(2 \geq q \geq 2n/(n+2l)$, if $n \geq 2l$, and $2 \geq q > 1$, if $n < 2l)$, as well as a necessary and sufficient condition for the discreteness of the spectrum of the operator of this problem in $L_2(\Omega)$. The method used is a development of the method applied by A. M. Molchanov ([1]) for proving criteria of discreteness of the spectrum of multidimensional elliptic operators of the second order.

The conditions given in the paper for the case $n \geq 2l$ are formulated in terms of the polyharmonic capacity defined below. The formulations of the results for $n < 2l$, for a reason that will become clear later, are elementary and do not use the concept of capacity. We include this case in the general scheme, but it can be investigated independently by the method of the paper of M. Sh. Birman and B. S. Pavlov ([2]).

The polyharmonic operator is chosen as a model—with insignificant changes all the arguments carry over to elliptic equations with variable coefficients.

Let Ω be an open subset of R_n. Denote by $\mathring{\mathcal{L}}_2^{(l)}(\Omega)$ the class of functions $u(x) \in C^{(l-1)}(R_n) \cap C^{(l)}(\Omega)$, equal to zero in $C\Omega$ and outside some ball, for which $\nabla_l u \in L_2(\Omega)$, $\nabla_l = \{(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}\}$, $\sum \alpha_i = l$, $\alpha_i \geq 0$. The closure of $\mathring{\mathcal{L}}_2^{(l)}(\Omega)$ in the norm $\|\nabla_l u\|_{L_2(\Omega)}$ generates the Hilbert space $\mathring{L}_2^{(l)}(\Omega)$.

Definition 1. Let G be an open bounded set in R_n and F a closed subset of G. The set $K = G \setminus F$ will be called an n-dimensional condenser.

Definition 2. Let K be an arbitrary condenser. We shall say that a function $u(x)$ belongs to the set $\mathfrak{A}(K)$ if: 1) $u(x) = 1$ on the set F and $u(x) = 0$ on the set CG; 2) $u(x) \in C^{(l-1)}(R_n) \cap L_2^{(l)}(R_n)$.

Definition 3. We shall call the number

$$c_l^{(n)}(K) = a_{n,l} \inf\left\{\int_K (\nabla_l u)^2\,dx;\ u(x) \in \mathfrak{A}(K)\right\}$$

the n-dimensional l-harmonic capacity of the condenser K.

Here $a_{n,l}$ is a certain constant, whose value is immaterial for us. For $n > 2l$ we denote by $\mathrm{cap}_l^{(n)}(F)$ the n-dimensional l-harmonic capacity of the compact set $F \in R_n$, i.e. $\inf\{c_l^{(n)}(G \setminus F);\ G \supset F\}$*. For l-times differenti–

$$\text{* For } n \leq 2l \quad \inf\{c_l^{(n)}(G \setminus F);\ G \supset F\} = 0.$$

For differentiable functions defined in the closed n-dimensional cube $\overline{Q}_d$ with edge length d, define the norm

$$\|u\|^2_{l,\overline{Q}_d} = \sum_{j=1}^{l} d^{2(j-l)} \|\nabla_j u\|^2_{L_2(\overline{Q}_d)}.$$

Let $F \subset \overline{Q}_d$. We introduce for consideration the class $\mathfrak{B}(F)$ of functions $u(x) \in C^{(l-1)}(\overline{Q}_d) \cap L_2^{(l)}(\overline{Q}_d)$ equal to zero on F, and denote by $b_{l,q}^{(n)}(F)$ the number*

$$\inf\left\{\|u\|^2_{l,\overline{Q}_d};\ \int_{\overline{Q}_d} |u|^q\,dx = d^n, \quad u \in \mathfrak{B}(F)\right\},$$

where $1 \leq q \leq 2n/(n-2l)$ for $n > 2l$ and $1 \leq q < \infty$ for $n \leq 2l$. In what follows γ, γ_i are positive constants depending only on the parameters n, l, q, j. By Q_{2d} we denote the open cube with edge length $2d$, whose center coincides with the center of the cube $\overline{Q}_d$ and whose faces are parallel to the faces of $\overline{Q}$. We also introduce the notation

$$\overline{f} = d^{-n} \int_{\overline{Q}_d} f\,dx.$$

Lemma 1. 1°. For any compact set $F \subset \overline{Q}_d$ the inequality

$$c_l^{(n)}(Q_{2d} \setminus F) \leq \gamma_1 b_{l,q}^{(n)}(F)$$

holds.

2°. If

$$c_l^{(n)}(Q_{2d} \setminus F) \leq \gamma d^{n-2l},$$

where γ is a sufficiently small constant, then

$$c_l^{(n)}(Q_{2d} \setminus F) \geq \gamma_2 b_{l,q}^{(n)}(F).$$

Lemma 2. For $n > 2l$, for any compact set $F \subset \overline{Q}_d$ the inequalities

$$\operatorname{cap}_l^{(n)} F \leq c_l^{(n)}(Q_{2d} \setminus F) \leq \gamma_3 \operatorname{cap}_l^{(n)} F$$

are valid.

It follows from Lemmas 1 and 2, incidentally, that for closed sets F situated in the cube $\overline{Q}_d$ and satisfying the condition

$$\operatorname{cap}_l^{(n)} F \leq \gamma\gamma_3^{-1} d^{\,n-2l} \quad (n > 2l),$$

the l-harmonic capacity is equivalent to the set function $e_{l,d}^n(F)$, defined by V. A. Kondrat' ev ([3]).** For harmonic capacity this fact, in essence, was proved in the paper of A. M. Molchanov ([1]).

Proof of Lemma 1. 1°. Let $n > 2l$. Obviously, it is enough to consider the case

$$q = q^* = \frac{2n}{n-2l}.$$

Using the integral representation of S. L. Sobolev (in the form proposed by L. V. Kantorovich ([4])) and S. L. Sobolev' s inequality ([5]), we obtain, for functions $u \in \mathfrak{B}(F)$, $F \subset \overline{Q}_d$, the estimate

$$\|u - \overline{u}\|_{L_{q^*}(\overline{Q}_d)} \leq \gamma_4 \|u\|_{l,\overline{Q}_d}.$$

Normalize the function $u(x)$ by the condition $\overline{|u|^{q^*}} = 1$ and assume, without loss of generality, that $\overline{u} \geq 0$. Then from the last inequality it follows that

$$1 - \overline{u} \leq \gamma_4 d^{\,l-n/2} \|u\|_{l,\overline{Q}_d}.$$

Introduce the notation $\varphi = 1 - u$. Since $\varphi \geq 0$, we have

$$\overline{\varphi}^{\,2} \leq \gamma_4^2 d^{\,2l-n} \|u\|_{l,\overline{Q}_d}^2.$$

Applying the following inequality, which follows from Poincaré' s inequality:

$$\|\varphi\|_{L_2(\overline{Q}_d)}^2 \leq \gamma_5 d^{2l} \|\varphi\|_{l,\overline{Q}_d}^2 + d^n \overline{\varphi}^{\,2},$$

we obtain the estimate

$$d^{-2l} \|\varphi\|_{L_2(\overline{Q}_d)}^2 \leq (\gamma_5 + \gamma_4^2) \|\varphi\|_{l,\overline{Q}_d}^2. \tag{1}$$

* The set function introduced by V. A. Kondrat' ev ([3]) is, obviously, equivalent to $b_{l,2}^{(n)}$.

** Without the smallness condition on the capacity, the equivalence of $e_{l,d}^{n}$ and $\mathrm{cap}_l^{(n)}$ is obviously not preserved.

Extend the function φ outside Q_d in such a way that the extension $\widetilde{\varphi}$ satisfies the inequality ([6])

$$\|\widetilde{\varphi}\|_{l,Q_{2d}} \leqslant \gamma_6 \|\varphi\|_{l,\overline{Q}_d}.$$

Denote by $\eta(x)$ a function in $C^{(l)}(R_n)$, equal to one in $\overline{Q}_d$, to zero outside Q_{2d}, and satisfying the condition $|\nabla^j \eta| \leqslant \gamma_7 d^{-j}$. Since $\widetilde{\varphi}\eta \in \mathfrak{A}(Q_{2d} \setminus F)$, we have

$$c_l^{(n)}(Q_{2d} \setminus F) \leqslant \gamma_8 \big(\|\widetilde{\varphi}\|^2_{l,Q_{2d}} + d^{-2l} \|\widetilde{\varphi}\|^2_{L_2(Q_{2d})}\big). \tag{2}$$

To estimate the second term we apply the inequality (see ([7]), p. 490)

$$\|\widetilde{\varphi}\|^2_{L_2(Q_{2d})} \leqslant \gamma_9 \big(d^{2l} \|\widetilde{\varphi}\|^2_{l,Q_{2d}} + \|\widetilde{\varphi}\|^2_{L_2(\overline{Q}_d)}\big).$$

Hence we derive

$$\|\widetilde{\varphi}\|^2_{L_2(Q_{2d})} \leqslant \gamma_9 \big(\gamma_6^2 d^{2l} \|\varphi\|^2_{l,\overline{Q}_d} + \|\varphi\|^2_{L_2(\overline{Q}_d)}\big).$$

From the last estimate and inequality (2) it follows that

$$c_l^{(n)}(Q_{2d} \setminus F) \leqslant \gamma_{10} \big(\|\varphi\|^2_{l,\overline{Q}_d} + d^{-2l} \|\varphi\|^2_{L_2(\overline{Q}_d)}\big).$$

By virtue of (1) and the equality $|\nabla^j \varphi| = |\nabla^j u|$, we finally obtain

$$c_l^{(n)}(Q_{2d} \setminus E) \leqslant \gamma_1 \|u\|^2_{l,\overline{Q}_d}.$$

The case $n \leqslant 2l$ is treated analogously.

2°. It suffices to carry out the proof for $q = 1$. Let $F \subset \overline{Q}_d$ be a compact set satisfying the condition $c_l^{(n)}(Q_{2d} \setminus F) \leqslant \gamma d^{n-2l}$, and suppose that, for the function $\psi \in \mathfrak{A}(Q_{2d} \setminus F)$,

$$a_{n,l} \|\nabla_l \psi\|^2_{L_2(Q_{2d})} \leqslant c_l^{(n)}(Q_{2d} \setminus F) + \varepsilon, \qquad \varepsilon > 0.$$

Put $u = 1 - \psi$. Then $|u| \geqslant \overline{u} = 1 - \psi$. We apply Friedrichs' inequality

$$\overline{\psi} \geqslant (\overline{\psi^2})^{1/2} \leqslant \gamma_{11} d^{\,l-n/2} \|\nabla_l \psi\|_{L_2(Q_{2d})} \leqslant a_{n,l}^{-1/2} \gamma_{11} d^{\,l-n/2} \big[c_l^{(n)}(Q_{2d} \setminus F) + \varepsilon\big]^{1/2}.$$

Passing to Fourier transforms, it is easy to prove the inequality

$$\int_{Q_{2d}} (\nabla_j \psi)^2 \, dx \leqslant \gamma_{12} \left\{ \int_{Q_{2d}} (\nabla_l \psi)^2 \, dx \right\}^{j/l} \left\{ \int_{Q_{2d}} \psi^2 \, dx \right\}^{1-j/l} \qquad (1 \leqslant j \leqslant l). \tag{3}$$

Using (3), we obtain

$$\|\psi\|_{l,Q_{2d}} \leqslant \gamma_{13} \|\nabla_l \psi\|_{L_2(Q_{2d})}.$$

We may assume that $\gamma < (2\gamma_{11})^{-2} a_{n,l}$. Then, for sufficiently small ε, $2|\overline{u}| \geqslant 1$. In addition,

$$\|u\|^2_{l,\overline{Q}_d} \leqslant \|\psi\|^2_{l,Q_{2d}} \leqslant \gamma_{13}^2 \|\nabla_l \psi\|^2_{L_2(Q_{2d})} \leqslant a_{n,l}^{-1} \varepsilon \gamma_{13}^2 [c_l^{(n)}(Q_{2d} \setminus F) + \varepsilon].$$

Consequently,

$$b_{l,1}^{(n)}(F) \leqslant 4 a_{n,l}^{-1} \gamma_{13}^2 c_l^{(n)}(Q_{2d} \setminus F).$$

We omit the proof of Lemma 2, since it is quite simple. With the aid of Lemma 1 one proves

Lemma 3. *In order that, for all $u \in L_2^{(l)}(\Omega)$, the inequality*

$$\|u\|_{L_q(\Omega)} \leqslant \varkappa \|\nabla_l u\|_{L_2(\Omega)} \tag{4}$$

hold,

where $q \in \left[2, \dfrac{2n}{n-2l}\right]$ for $n > 2l$ and $q \in [2, \infty)$ for $n \leqslant 2l$, it is necessary and sufficient that, for some $d > 0$, the inequality

$$\inf_{Q_d \subset R_n} c_l^{(n)} \left[Q_{2d} \setminus (\overline{Q}_d \cap C\Omega) \right] > 0. \tag{5}$$

hold.

If $2l > n$, then for an arbitrary point $P \in \overline{Q}_d$

$$c_l^{(n)}(Q_{2d} \setminus P) \leqslant \gamma_0 d^{\,n-2l}.$$

Hence, for $2l > n$, condition (5) is fulfilled if and only if, for some $d > 0$, for all cubes $\overline{Q}_d$ the set $\overline{Q}_d \cap C\Omega$ is nonempty.

From Lemmas 2 and 3, and also from what has just been said about the case $2l > n$, it follows that

Theorem 1. For the inequality (4) to hold for all $u \in \mathring{L}_2^{(l)}(\Omega)$, it is necessary and sufficient that the following conditions be satisfied:

1. For some $d > 0$,

$$\inf\{\,\mathrm{cap}_l^{(n)}(C\Omega \cap \overline{Q}_d);\ \overline{Q}_d \cap R_n\,\} > 0,$$

   if $n > 2l$.

2. For some $d > 0$,

$$\inf\{\,c_l^{(n)}[\,Q_{2d} \setminus (\overline{Q}_d \cap C\Omega)\,];\ \overline{Q}_d \subset R_n\,\} > 0,$$

   if $n = 2l$.

3. The set Ω contains no arbitrarily large cubes, if $n < 2l$.

Let us also note that if the complement of the set Ω is connected, then the last condition is necessary and sufficient also in the case $n = 2$ (cf. ([1], [3])).

Theorem 1 admits the following equivalent formulation.

Theorem 1′. The conditions of Theorem 1 are necessary and sufficient for the existence and uniqueness in $\mathring{L}_2^{(l)}(\Omega)$ of a solution of the equation $\Delta^l u = f$ for all $f \in L_q(\Omega)$ $(2 \geqslant q > 2n/(n+2l)$ for $n \geqslant 2l$; $2 \geqslant q > 1$ for $n < 2l)$.

The following theorem is a generalization of the criterion for discreteness of the spectrum of the first boundary-value problem for the Laplace operator found by A. M. Molchanov ([1]).

Theorem 2. For discreteness of the spectrum of the operator Δ^l, defined in $\mathring{L}_2^{(l)}(\Omega) \cap L_2(\Omega)$, it is necessary and sufficient that the following conditions be satisfied:

1. For every $d > 0$,

$$\liminf_{R\to\infty}\{\,d^{2l-n}\,\mathrm{cap}_l^{(n)}(\overline{Q}_d \cap C\Omega);\ \overline{Q}_d \subset CS_R\,\} > c > 0,$$

   if $n > 2l$. Here S_R is the ball of radius R centered at the origin, and the constant c does not depend on d.

2. For every $d > 0$,

$$\liminf_{R\to\infty}\{\,c_l^{(n)}[\,Q_{2d} \setminus (\overline{Q}_d \cap C\Omega)\,];\ \overline{Q}_d \subset CS_R\,\} > c > 0,$$

   if $n = 2l$.

3. The set Ω contains no sequence of pairwise disjoint equal cubes, if $n < 2l$.

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Note: Figure translations are in progress. See original paper for figures.