ON A VARIATIONAL METHOD FOR SOLVING A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

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Abstract

Full Text

ON A VARIATIONAL METHOD FOR SOLVING A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

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1. In the preceding note ([1]) we indicated a class of weighted embedding theorems that admit a complete converse. In this connection, the results of Theorems 8 and 9 ([1]) also carry over to bounded domains. In the present note we intend to supplement the preceding results and to indicate a class of well-posed boundary-value problems for a new type of degenerate elliptic equations.

For simplicity of exposition, we restrict ourselves to considering the half-space $\overset{+}{R}_2$ in the two-dimensional case. Consider the set of functions $f(x, y)$, finite in the half-space $x > 0$ with the manifold $x = 0$ removed (see the definition in ([1])), and define on it the differential operator

$$D_x f = \frac{1}{x}\frac{\partial f(x,y)}{\partial x} \tag{1}$$

and the powers of this operator

$$D_x^l f = \frac{\partial}{x\partial x}\left(\frac{\partial^{l-1} f(x,y)}{(x\partial x)^{l-1}}\right) \quad (l = 2, 3, \ldots). \tag{2}$$

Define the functional space $W^{(l)}_{x,2,\gamma}(\overset{+}{R}_2)$ as the closure of the set of functions $f(x, y)$, finite in the half-space $\overset{+}{R}_2$ with the manifold $x = 0$ removed, with respect to the norm

$$\|f\|^2_{W^{(l)}_{x,2,\gamma}(\overset{+}{R}_2)} = \int_{\overset{+}{R}_2} |f(x,y)|^2 x^{2\gamma}\,dx\,dy + \sum_{k=1}^{l}\int_{\overset{+}{R}_2}\left|x^k\frac{\partial^k f(x,y)}{(x\partial x)^k}\right|^2 x^{2\gamma}\,dx\,dy, \tag{3}$$

where γ is a positive number.

We define the functional space $W^{(l_1)}_{y,2,\gamma}(\overset{+}{R}_2)$ as the closure of the set of functions $f(x, y)$, finite in the half-space $\overset{+}{R}_2$ with the manifold $x = 0$ removed, with respect to the norm

$$\|f\|^2_{W^{(l_1)}_{y,2,\gamma}(\overset{+}{R}_2)} = \int_{\overset{+}{R}_2} |f(x,y)|^2 x^{2\gamma}\, dx\, dy + \sum_{k=1}^{l_1'} \int_{\overset{+}{R}_2} \left|\frac{\partial^k f(x,y)}{\partial y^k}\right|^2 x^{2\gamma}\, dx\, dy + \sum_{k=1}^{l_1'} \mathcal{L}^{(2)}_{k,\gamma}\left(\frac{\partial^k f(x,y)}{\partial y^k}\right), \tag{4}$$

where by $\mathcal{L}^{(2)}_{k,\gamma}(\partial^k f/\partial y^k)$ one should understand the expression

$$\mathcal{L}^{(2)}_{k,\gamma}\left(\frac{\partial^k f}{\partial y^k}\right) = \int_{-\infty}^{\infty}\int_{\overset{+}{R}_2} \frac{\left|\partial^k f(x,y+h)/\partial y^k - \partial^k f(x,y)/\partial y^k\right|^2}{|h|^{1+2\lambda}} x^{2\gamma}\, dx\, dy\, dh, \tag{5}$$

and here $l_1 = l_1' + \lambda$ and $0 < \lambda < 1$.

We define the functional space $W^{(l,l_1)}_{x,y,2,\gamma}(\overset{+}{R}_2)$ as the intersection of the functional classes $W^{(l)}_{x,2,\gamma}(\overset{+}{R}_2)$ and $W^{(l_1)}_{y,2,\gamma}(\overset{+}{R}_2)$, and introduce the norm, as usual, by the formula

$$\|f\|^2_{W^{(l,l_1)}_{x,y,2,\gamma}(\overset{+}{R}_2)} = \|f\|^2_{W^{(l)}_{x,2,\gamma}(\overset{+}{R}_2)} + \|f\|^2_{W^{(l_1)}_{y,2,\gamma}(\overset{+}{R}_2)}. \tag{6}$$

The case $\gamma = 0$ is contained in the preceding note ([1]). As in the preceding case ([1]), instead of the indicated norms one may consider norms equivalent to them. Here the index l assumes only positive integer values.

Theorem 1. Let $f \in W^{(l,l_1)}_{x,y,2,\gamma}(\overset{+}{R}_2)$, where $\gamma > 0$, and let, for nonnegative integers k, the inequality

$$\mu = \mu(k,\gamma) = 1 - 2k/l - (2\gamma+1)/2l > 0 \tag{7}$$

hold. Then the derivatives $\partial^k f(x,y)/(x\partial x)^k$, for all those y for which they are quadratically summable over R_1 with respect to x, belong to the space $W^{(\bar{l}_1)}_{y,2}(R_1)$ with $\bar{l}_1 = \mu l_1$. Moreover, the inequality

$$\left\|\left.\frac{\partial^k f}{(x\partial x)^k}\right|_{x=0}\right\|_{W^{(\bar{l}_1)}_{y,2}(R_1)} \le c\|f\|_{W^{(l,l_1)}_{x,y,2,\gamma}(\overset{+}{R}_2)} \tag{8}$$

holds, where the constant c does not depend on f.

Remark 1. $W^{(\bar{l}_1)}_{y,2}(R_1)$ denotes the usual Aronszajn–Slobodeckii space ([2]). The converse assertion is also true.

Theorem 2. Let nonnegative integers k be given, for which the inequality

$$\mu = \mu(k,\gamma) = 1 - 2k/l - (2\gamma+1)/2l > 0 \tag{9}$$

holds. For such k, let functions $\varphi^{(k)}(y) \in W_{y,2}^{(\bar{l}_1)}(R_1)$ with $\bar{l}_1 = \mu l_1$ be prescribed on R_1. There exists a function $\bar{f}(x,y) \in W_{x,y,2,\gamma}^{(l,l_1)}(\overset{+}{R}_2)$ such that

$$\lim_{x\to+0}\left\| \frac{\partial^k \bar{f}(x,y)}{(x\partial x)^k} - \varphi^{(k)}(y)\right\|_{W_{y,2}^{(\bar{l}_1)}(R_1)} = 0 \tag{10}$$

for all admissible k. Moreover, the inequality

$$\|\bar{f}\|_{W_{x,y,2,\gamma}^{(l,l_1)}(\overset{+}{R}_2)} \le c\sum_k \|\varphi^{(k)}(y)\|_{W_{y,2}^{(\bar{l}_1)}(R_1)} \tag{11}$$

holds with a constant c independent of $\varphi^{(k)}$.

Let us note that the theorems stated here remain valid if, instead of nonnegative integers k satisfying the inequality

$$\mu = \mu(k,\gamma) = 1 - 2k/l - (2\gamma+1)/2l > 0, \tag{12}$$

one simply takes nonnegative numbers k satisfying the same inequality. What is meant by this is clear from the theorems given below.

Theorem 3. Let $f \in W_{x,y,2,\gamma}^{(l,l_1)}(\overset{+}{R}_2)$ $(\gamma > 0)$, and let, for nonnegative numbers k, the inequality

$$\mu = \mu(k,\gamma) = 1 - 2k/l - (2\gamma+1)/2l > 0 \tag{13}$$

hold. Then the derivatives $\widetilde{\mathcal{D}}_x^k f$, for all those y for which they are quadratically summable over R_1 with respect to x, belong to the space $W_{y,2}^{(\bar{l}_1)}(R_1)$ with $\bar{l}_1 = \mu l_1$. Moreover, the inequality

$$\left\|\widetilde{\mathcal{D}}_x^k f\Big|_{x=0}\right\|_{W_{y,2}^{(\bar{l}_1)}(R_1)} \le c\|f\|_{W_{x,y,2,\gamma}^{(l,l_1)}(\overset{+}{R}_2)} \tag{14}$$

holds with a constant c independent of f.

Here and in the following theorem, in the case of an integer k, $\widetilde{\mathcal{D}}_x^k f$ should be understood as the derivative $\partial^k f/(x\partial x)^k$, while in the case of fractional k, $\widetilde{\mathcal{D}}_x^k f$ should be understood as the integro-differential operator ($k = \bar{k} + \beta/2$, $\bar{k}$ a nonnegative integer and $0 < \beta/2 < 1$, with $\beta < \gamma$)

$$\frac{\partial^{\bar{k}+\beta/2} f}{x^{\bar{k}-\beta/2}\partial x^{\bar{k}+\beta/2}} =$$

$$= \frac{1}{x^{2\bar{k}-\beta+2\gamma}}\frac{\partial}{\partial x}\left[\frac{1}{\Gamma(1-\beta/2)}\int_0^x (x^2-\tau^2)^{-\beta/2}\tau^{\bar{k}}\left(\tau^{\bar{k}}\frac{\partial^k f(\tau,y)}{(\tau\partial\tau)^k}\right)\tau^{2\gamma}\,d\tau\right]. \tag{15}$$

For the definition of the integro-differential operator (15), see ([1]). The converse theorem is also true.

Theorem 4. Let nonnegative numbers k be given, for which inequality (13) is satisfied. For such numbers k let us prescribe on R_1 functions $\varphi^{(k)}(y) \in W_{y,2}^{(\bar{l}_1)}(R)$, where $\bar{l}_1 = \mu l_1$. There exists an $\bar{f} \in W_{x,y,2,\gamma}^{(l,l_1)}(\overset{+}{R}_2)$ such that

$$\lim_{x \to +0} \left\| \widetilde{\mathcal{D}}_x^k \bar{f}(x,y) - \varphi^{(k)}(y) \right\|_{W_{y,2}^{(\bar{l}_1)}(R)} = 0. \tag{16}$$

for all admissible k. Moreover, the inequality

$$\|f\|_{W_{x,y,2,\gamma}^{(l,l_1)}(\overset{+}{R}_2)} \leq c \sum_k \left\|\varphi^{(k)}(y)\right\|_{W_{y,2}^{(\bar{l}_1)}(R_1)} \tag{17}$$

holds, with a constant c independent of $\varphi^{(k)}$.

It is interesting to note that many properties of the fractional differentiation operator previously studied by the author (3–7) carry over to integro-differential operators of the type of operator (15), which here plays the role of a boundary operator.

2. We shall consider the equation

$$\mathcal{L}(f) = \Delta^2 f + \frac{c_1 + 2(3+2\gamma)}{x} \frac{\partial^3 f}{\partial x^3} + \frac{4\gamma}{x} \frac{\partial^3 f}{\partial x\, \partial y^2} +$$

$$+ \frac{c_2 + \tilde{c}_1\, 2(3+2\gamma) + (3+2\gamma)(1+2\gamma)}{x^2} \frac{\partial^2 f}{\partial x^2} +$$

$$+ \frac{c_3 + 2\tilde{c}_2(3+2\gamma) - (3+2\gamma)(1+2\gamma)}{x^3} \frac{\partial f}{\partial x} + f = 0, \tag{18}$$

which is the Euler equation for the functional

$$\widetilde{\mathcal{D}}_\gamma^{(2)}(f) = \int_{\overset{+}{R}_2} |f|^2 x^{2\gamma}\, dx\, dy +$$

$$+ \int_{\overset{+}{R}_2} \left\{ \left[x^2 \frac{\partial^2 f}{(x\, \partial x)^2} \right]^2 + 2 \left[x \frac{\partial^2 f}{x\, \partial x\, \partial y} \right]^2 + \left[\frac{\partial^2 f}{\partial y^2} \right]^2 \right\} x^{2\gamma}\, dx\, dy. \tag{19}$$

(the constants $c_1, c_2, c_3, \tilde{c}_1, \tilde{c}_2$ have quite definite values). The functional (19), as is not hard to see, is a norm in the space $W_{x,y,2,\gamma}^{(2,2)}(\overset{+}{R}_2)$. The principal part of equation (18) can be written in self-adjoint form, and it will have the form

$$\mathcal{L}(f) = \frac{\partial^2}{\partial x^2}\left(x^3 \frac{\partial^2 f}{\partial x^2} \right) + 2 \frac{\partial^2}{\partial x\, \partial y}\left(x^3 \frac{\partial^2 f}{\partial x\, \partial y} \right) + \frac{\partial^2}{\partial y^2}\left(x^3 \frac{\partial^2 f}{\partial y^2} \right) +$$

$$+x^2 c_\gamma^{(1)} \frac{\partial^3 f}{\partial x\, \partial y^2} + x^2 c_\gamma^{(2)} \frac{\partial^3 f}{\partial x^3} + x c_\gamma^{(3)} \frac{\partial^2 f}{\partial x^2} + c_\gamma^{(4)} \frac{\partial f}{\partial x} + x^3 f = 0. \tag{20}$$

Statement of the problem. It is required to find a generalized solution of equation (20), $f(x, y)$, taking on the boundary of the domain $x = 0$ the prescribed value $\varphi(y)$.

We shall agree to call a function $\varphi(y)$, prescribed on the boundary of the domain $\overset{+}{R}_2$, **admissible** if there exists a function $f \in W^{(2,2)}_{x,y,2,\gamma}(\overset{+}{R}_2)$ for which $\varphi(y)$ serves as the boundary value. Denote by $W^{(2,2)}_{x,y,2,\gamma}(\varphi)$ the set of functions $f \in W^{(2,2)}_{x,y,2,\gamma}(\overset{+}{R}_2)$ such that $f\big|_{x=0} = \varphi$. From item 1 it follows that $W^{(2,2)}_{x,y,2,\gamma}(\varphi)$ is a nonempty set. For every $f \in W^{(2,2)}_{x,y,2,\gamma}(\varphi)$ we have $0 \le \widetilde{\mathcal{D}}^{(2)}_\gamma(f) < \infty$. Therefore there exists an exact lower bound of the values $\widetilde{\mathcal{D}}^{(2)}_\gamma(f)$:

$$d = \inf_{f \in W^{(2,2)}_{x,y,2,\gamma}(\varphi)} \widetilde{\mathcal{D}}^{(2)}_\gamma(f). \tag{21}$$

Obviously, from the set $W^{(2,2)}_{x,y,2,\gamma}(\varphi)$ one can extract a sequence $\{f_k\}$ for which

$$\lim_{k\to\infty} \widetilde{\mathcal{D}}^{(2)}_\gamma(f_k) = d. \tag{22}$$

The sequence $\{f_k\}$ is called **minimizing**.

Theorem 5. The minimizing sequence of functions $\{f_k\}$ converges in $W^{(2,2)}_{x,y,2,\gamma}(\overset{+}{R}_2)$ to a certain function $f_0(x, y)$. This function belongs to $W^{(2,2)}_{x,y,2,\gamma}(\varphi)$ and gives the functional $\widetilde{\mathfrak{D}}^{(2)}_\gamma(f)$ its least value.

Theorem 6. The function f_0, giving the minimum to the functional $\widetilde{\mathfrak{D}}^{(2)}_\gamma(f)$ in $W^{(2,2)}_{x,y,2,\gamma}(\varphi)$, is a generalized solution of equation (20) under the boundary condition

$$f\big|_{x=0} = \varphi. \tag{23}$$

Remark 2. We call the function f_0 a generalized solution of equation (20) under the boundary condition (23) if, for every function z, finite in the half-space $\overset{+}{R}_2$ with the manifold $x = 0$ removed, the relation

$$\widetilde{\mathfrak{D}}^{(2)}_\gamma(f_0, z) = \int_{\overset{+}{R}_2} f_0 z x^{2\gamma}\, dx\, dy +$$

$$+\int_{\overset{+}{R}_2}\left\{\left[x^2\frac{\partial^2 f_0}{(x\partial x)^2}x^2\frac{\partial^2 z}{(x\partial x)^2}\right]+2\left[x\frac{\partial^2 f_0}{x\partial x\,\partial y}x\frac{\partial^2 z}{x\partial x\,\partial y}\right]+\frac{\partial^2 f_0}{\partial y^2}\frac{\partial^2 z}{\partial y^2}\right\}x^{2\gamma}\,dx\,dy=0. \tag{24}$$

By the very definition of a generalized solution, it is unique. From the preceding theorems of Sections 1 and 2 it follows:

Theorem 7. In order that the problem of finding a generalized solution of equation (20) under the boundary condition (23) be solvable in $W^{(2,2)}_{x,y,2,\gamma}(\overset{+}{R}_2)$, it is necessary and sufficient that $\varphi \in W^{(\bar{l}_1)}_{y,2}(R_1)$ with $\bar{l}_1 = 3/2-\gamma$.

For the solution f_0 the two-sided estimate is valid

$$c_2\|\varphi\|_{W^{(\bar{l}_1)}_{y,2}(R_1)} \le \|f_0\|_{W^{(2,2)}_{x,y,2,\gamma}(\overset{+}{R}_2)} \le c_1\|\varphi\|_{W^{(\bar{l}_1)}_{y,2}(R_1)}. \tag{25}$$

If in equation (20) the number γ is different from zero, then the necessary and sufficient condition for solvability of problem (20)–(23) can also be formulated in terms of the integro-differential operator (15). Everywhere in Theorems 5, 6, and 7, the boundary condition $f|_{x=0} = \varphi$, on the basis of Theorems 3 and 4 of Section 1, may be replaced by the boundary condition

$$x^{\beta/2}\partial^{\beta/2} f/\partial x^{\beta/2}\big|_{x=0} = \varphi_1 \qquad (\beta<\gamma). \tag{26}$$

Then Theorem 7 takes the following form:

Theorem 8. In order that the problem of finding a generalized solution of equation (20) under the boundary condition (26) be solvable in $W^{(2,2)}_{x,y,2,\gamma}(\overset{+}{R}_2)$, it is necessary and sufficient that $\varphi_1 \in W^{(\bar{l}_1)}_{y,2}(R_1)$ with $\bar{l}_1 = 3/2-\gamma-\beta$. For the solution f_0 the two-sided estimate holds

$$\bar{c}_2\|\varphi_1\|_{W^{(\bar{l}_1)}_{y,2}(R_1)} \le \|f_0\|_{W^{(2,2)}_{x,y,2,\gamma}(\overset{+}{R}{}^2)} \le \bar{c}_1\|\varphi_1\|_{W^{(\bar{l}_1)}_{y,2}(R_1)}. \tag{27}$$

In a completely analogous way one may consider the case of equations of order $2m$, in which the order of degeneration will be correspondingly equal to $2m-1$. Similar results are also valid for bounded domains in R_n.

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Note added in proof. At the present time the author has obtained a priori estimates for solutions of certain classes of degenerating equations with variable coefficients.

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Note: Figure translations are in progress. See original paper for figures.