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Abstract

Full Text

MATHEMATICS

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ON THE EMBEDDING OF MODULES

By an A-module, where A is an associative ring, we shall mean a right A-module. The annihilator of an A-module M is the set

$$(0 : M)_A = \{a \mid a \in A,\ Ma = 0\}.$$

If $(0 : M)_A = 0$, then M is called **faithful**; if $(0 : M)_A = A$, i.e. $MA = 0$, then M is called **trivial**. A nontrivial A-module M is called a **divisible A-module** (see ([1]), p. 164) if the equation $x = ya$ is solvable in M for all $x \in M$ and all $a \notin (0 : M)_A$ from A.

Lemma 1. *Let A be an arbitrary ring without zero divisors and with identity. Every injective A-module is a faithful divisible A-module (cf. ([1]), p. 164).*

Lemma 2 ([1]). *Every unitary A-module, where A is an arbitrary ring with identity, is isomorphically embeddable in some injective A-module M.*

Recall that a nontrivial A-module M is called **primary** ([2]) if from the equality $xB = 0$, where $x \in M$, B is an ideal of the ring A, it follows that $x = 0$ or $B \subseteq (0 : M)_A$.

Lemma 3. *Let the faithful primary A-module M be a submodule of the A-module N. Then N is mapped homomorphically onto some primary A-module N' containing a submodule isomorphic to the module M.*

Proof. Denote by N_0 the zero submodule of the A-module N. If for all $\beta < \alpha$ we have already constructed A-modules N_β, then set, by definition,

$$N_\alpha = \{z \mid zB \subseteq N_\beta \text{ for some } \beta < \alpha \text{ and some nonzero ideal } B \text{ of } A\}.$$

It is clear that N_α is a submodule in N, and $N_\alpha \supseteq N_\beta$ for all $\beta < \alpha$. From the faithful primariness of the A-module M and the construction of N_α it follows that $M \cap N_\alpha = 0$ for every α. There exists an $\alpha = \tau$ such that $N_\tau = N_{\tau+1}$. Denote N_τ by M' and consider the factor A-module $N' = N/M'$. In view of the construction of M', it is clear that N' is a primary A-module. Moreover, since $M \cap M' = 0$, N' contains as a submodule the A-module

$$M + M'/M' \simeq M/M \cap M' = M/0 = M.$$

Recall that a ring A is called **primary** if the product of any two nonzero ideals of A is a nonzero ideal. A class of primary rings L is called **special** ([3]) if every nonzero ideal and every primary extension of a ring from L belong to the class L. The classes of all primary rings, of all rings without zero divisors, and of all primitive rings are special classes of rings.

Lemma 4. *Every ring A from a special class of rings L is isomorphically embeddable as an ideal in some ring with identity A' from the same class L.*

Indeed, embed the ring A in the usual way into a ring with identity

$$A'' = A + C,$$

where C is the ring of integers. Let A^* be the annihilator of the ring A in the ring A'', i.e.

$$A^* = \{a \mid a \in A'',\ Aa = aA = 0\}.$$

Since A is a primary ring, it is easy to see that $A^* \cap A = 0$ and the factor ring A''/A^* is a primary ring. But A''/A^* is an extension of the ring

$$A + A^*/A^* \simeq A \in L.$$

Consequently, the ring with identity

$$A' = A''/A^*$$

belongs to the special class of rings L.

We note that Lemma 4 generalizes known results on the embedding of a ring without zero divisors in a ring with identity without zero divisors, on the embedding of a primary ring in a primary ring with identity, and so on.

Proposition 1. Let A be an arbitrary integral domain, p the characteristic of the ring A, and C_p the field of fractions of the residue ring of the integers modulo p.* Every faithful primary A-module M is embedded in some faithful primary divisible A-module M', which is at the same time also a faithful divisible C_p-module.

Proof. By means of Lemma 4 we embed the ring A in the ring $A_1 = A + C/A^*$, without zero divisors and with identity. Defining, by definition, $x(a + n \cdot 1) = xa + nx$, we turn M into a unitary $A + C$-module. Since $M(A^*A) = 0 = (MA^*)A$, it follows, by primarity of the A-module M, that $MA^* = 0$. Therefore, putting $x(b + A^*) = xb$, where $b \in A + C$, we turn M into a unitary A_1-module. Applying Lemma 2, we embed M in an injective A_1-module M_1, which, according to Lemma 1, is a faithful divisible A_1-module. Let us now note that the ring A_1 has the same characteristic as the ring A. Indeed, A_1 is an extension of the ring A, and A_1 contains no zero divisors. Since M is a faithful divisible A_1-module, for every $n \not\equiv 0 \pmod p$ the equation $x = y \cdot ne$, where e is the identity of the ring A_1, is solvable. But M_1 is a unitary A_1-module, and therefore $x = y \cdot ne = nye = ny$, i.e. M_1 is at the same time also a faithful divisible C_p-module. Since A is a subring of the ring A_1, M_1 is a faithful divisible A-module. By means of Lemma 3 we find the desired primary A-module M'.

Recall that a nontrivial A-module M is called an A-**module without zero divisors** if from $xa = 0$, where $x \in M$, $a \in A$, it follows that $x = 0$ or $a \in (0 : M)_A$ ([4]).

Let us note that every A-module without zero divisors is a primary A-module, and the ring A is an integral domain if and only if it has a faithful A-module without zero divisors.

From Proposition 1 and the preceding remark it follows that

Theorem 1. Every faithful A-module M without zero divisors is embedded in some primary divisible faithful A-module M', which is at the same time also a divisible faithful C_p-module, where p is the characteristic of the ring A.

Remark 1. In the case of the well-known example of A. I. Mal' tsev ([5]), not every integral domain is embeddable in a field. However, if such a ring A is regarded as an A-module, then, according to Theorem 1, it is embedded in some faithful primary divisible A-module.

To each associative ring A we assign some class Σ_A of primary A-modules, and let the class $\Sigma = \{\Sigma_A \mid A \in K\}$, where K is the class of all associative rings.

Recall that the class Σ is called a **special class** of modules if the following conditions are satisfied:

I. $M \in \Sigma_A,\ \bar{A} = A/P,\ MP = 0$, then under the composition $x\bar{a} = xa$ $M \in \Sigma_{\bar{A}}$.

II. If $M \in \Sigma_{\bar{A}}$, $\bar{A} = A/P$ is an arbitrary homomorphic image of the ring A, then under the composition $xa = x\bar{a}$ $M \in \Sigma_A$.

III. If $M \in \Sigma_A,\ MB \neq 0$, where B is an ideal in A, then $M \in \Sigma_B$.

IV. If $M \in \Sigma_B$, where B is an ideal of the ring A, then $MB \in \Sigma_A$.

The classes of all irreducible modules, all primary modules, and all modules without zero divisors are special classes of modules.

Let Σ be a special class of modules. Denote by $L(\Sigma)$ the class of all those rings A for which there exist faithful A-modules from the class Σ_A. It can be shown that $L(\Sigma)$ is a special class of rings ([4]).

Theorem 2. The class Σ of all primary divisible modules is a special class of modules. The special class of rings $L(\Sigma)$ coincides with the class of all integral domains.

Proof. Conditions I, II, III for the class Σ follow immediately

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* p will be either zero or a prime number.

from the fact that the class of all primary modules is a special class of modules. Let now M be a primary divisible B-module, where B is an ideal of the ring A. Since M is a primary B-module, MB will also be a primary A-module.

By the divisibility of the module M we obtain that $MB = M$. Therefore, if $a \notin (O : M)_A$, then $Ba \not\subset (O : M)_B$ and, since M is a divisible B-module, $Ma = MBa = M$, i.e., M will also be a divisible A-module. Thus condition IV is fulfilled and the class Σ is a special class of modules.

By Remark 1, every ring A without zero divisors has a faithful primary divisible A-module, i.e., $A \in L(\Sigma)$. Suppose now that A has a faithful divisible A-module M, and let $a \neq 0$, $b \neq 0$ be arbitrary elements of the ring A. Then for every $x \neq 0$ in M one can find $y, z \in M$ such that $x = yb$, $y = za$. Consequently, $x = zab \neq 0$, whence $ab \neq 0$. Thus the ring A has no zero divisors. The theorem is proved.

Remark 2. From Theorem 2 it follows easily that two different classes of modules may determine one and the same special class of rings. Indeed, if the ring A, considered as an A-module, is a faithful divisible A-module, then A, obviously, is a field. Meanwhile, not every ring without zero divisors is a field. Therefore the special classes of all primary divisible modules and of all modules without zero divisors are distinct, but determine one and the same special class of rings—the class of all rings without zero divisors.

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Note: Figure translations are in progress. See original paper for figures.