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Abstract

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MATHEMATICS

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ON A FINITE-DIFFERENCE METHOD FOR SOLVING THE TWO-PHASE STEFAN PROBLEM FOR A QUASILINEAR EQUATION

(Presented by Academician A. A. Dorodnitsyn, 9 IV 1963)

1°. In the one-dimensional case, the two-phase Stefan problem for a quasilinear equation with one phase-separation line and boundary conditions of the second kind can be formulated as follows ([1]): find functions $\bar{u}(x,t) \geq 0$, $u(x,t) \leq 0$, and $y(t)$ satisfying the relations:

$$\bar{u}_t = \bar{a}(\bar{u})\bar{u}_{xx} \quad \text{in } D_{\mathrm{II}} = \{0 < x < y(t),\, 0 < t < T\}; \tag{1}$$

$$u_t = a(u)u_{xx} \quad \text{in } D_{\mathrm{I}} = \{y(t) < x < l,\, 0 < t < T\}; \tag{2}$$

$$\bar{u}_x(0,t) = -\bar{q}(t), \quad 0 < t < T; \tag{3}$$

$$u_x(l,t) = -q(t), \quad 0 < t < T; \qquad u(x,0) = \varphi(x),\ 0 \leq x \leq l; \tag{4}$$

$$\bar{u}(y(t),t) = u(y(t),t) = 0, \quad 0 \leq t \leq T; \tag{5}$$

$$\nu y'(t) = u_x(y(t),t) - \bar{u}_x(y(t),t), \quad 0 < t < T, \qquad y(0) = 0, \tag{6}$$

where $\nu = \text{const} > 0$, $\bar{a}(\bar{u})$, $a(u)$ are physical characteristics of the substance, and T is the time of complete phase transformation, i.e. $y(T) = l^*$.

In ([1]), to prove the existence of a generalized solution of the first boundary-value Stefan problem in a very general formulation, an explicit difference scheme was used; however, for the practical computation of an approximate solution it is rather inefficient. We also mention the papers ([2–4]), where other methods are considered for the approximate solution of the two-phase Stefan problem for the linear heat-conduction equation.

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In the present note we prove the existence and uniqueness of the solution of problem (1)–(6) in the sense indicated below, under certain restrictions on the parameters of the problem; to find an approximate solution of the stated problem, we propose an effective implicit difference scheme (Theorem 1) and justify the convergence of the approximate solution obtained by this scheme to the solution of problem (1)–(6) (Theorem 2). We use the methodology developed in ([5,6]). Below we shall everywhere assume that $q(t) \geq 0$, $\bar{q}(t) - q(t) \geq \mathrm{const} > 0$, $\varphi(0) = 0$.

Definition. We shall call functions $\bar{u}(x,t)$, $u(x,t)$, $y(t)$ a **solution of problem** (1)–(6) if: 1) $y(t)$ is continuously differentiable for $0 \leq t \leq T$; $0 \leq y(t) < l$ for $0 \leq t < T$; $y(T) = l$; $y'(t) > 0$; 2) $\bar{u}(x,t)$ is defined and continuous in $\bar{D}_{\mathrm{II}} = \{0 \leq x \leq y(t),\, 0 \leq t \leq T\}$ together with the derivative $\bar{u}_x(x,t)$, has continuous in D_{II} and bounded in $\bar{D}_{\mathrm{II}}$ derivatives $\bar{u}_{xx}(x,t)$, $\bar{u}_t(x,t)$; the function $u(x,t)$ has analogous properties in $\bar{D}_{\mathrm{I}} = \{y(t) \leq x \leq l;\, 0 \leq t \leq T\}$; 3) all conditions (1)–(6) are satisfied.

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* Problems analogous to those considered in ([6]) and in the present note, in a more general formulation, are set out in detail in the collection of papers of the Computing Center of Moscow State University, *Numerical Methods in Gas Dynamics*, issue 3, Moscow State University Press (in press).

2°. Divide the segment $0 \leq x \leq l$ by points x_i into N equal parts with step h, where $x_i = ih$, $i = 1, 2, \ldots, N$, $x_N = l$. We shall take the time step τ_n ($n = 1, 2, \ldots$) to depend on n in such a way that for each $t = t_n = \sum_{k=1}^{n} \tau_k$ the broken line approximating the curve $y(t)$ falls on a node with coordinates ($y_n = nh, t_n$), moving at each time step by the amount h, i.e. $y_n - y_{n-1} = h$, where y_n is the approximate value of the phase-interface boundary at the instant $t = t_n$. We replace problem (1)–(6) by the following difference problem for determining τ_n and the approximate values $\bar{w}_{in}, w_{in}$ of the functions $\bar{u}(x,t), u(x,t)$:

$$\bar{a}(\bar{w}_{i,n-1})\, \delta_{xx} \bar{w}_{in} - \delta_{\bar{t}} \bar{w}_{in} = 0, \qquad 1 \leq i \leq n-1; \quad n = 2, 3, \ldots, N; \tag{7}$$

$$a(w_{i,n-1})\, \delta_{xx} w_{in} - \delta_{\bar{t}} w_{in} = 0, \qquad n+1 \leq i \leq N-1; \quad n = 1, 2, \ldots, N-2; \tag{8}$$

$$\delta_x \bar{w}_{0n} = -\bar{q}_{n-1}, \qquad n = 1, 2, \ldots, N; \tag{9}$$

$$\delta_x w_{N-1,n} = -q_{n-1}, \qquad 1 \leq n \leq N-1; \qquad w_{i0} = \varphi_i, \quad 0 \leq i \leq N; \tag{10}$$

$$\bar{w}_{nn} = w_{nn} = 0, \qquad 0 \leq n \leq N; \tag{11}$$

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$$\nu\,\frac{h}{\tau_n} = \delta_x w_{nn} - \delta_x \bar{w}_{n-1,n}, \qquad n = 1, 2, \ldots, N-1, \tag{12}$$

where the following notation has been adopted:

$$\delta_x z_{in} = \frac{1}{h}(z_{i+1,n} - z_{in}), \qquad \delta_{xx} z_{in} = \frac{1}{h^2}(z_{i+1,n} - 2z_{in} + z_{i-1,n}),$$

$$\delta_{\bar{t}} z_{in} = \frac{1}{\tau_n}(z_{in} - z_{i,n-1});$$

$$y_n = nh; \qquad t_n = \sum_{k=1}^{n} \tau_k; \qquad q_{n-1} = q(t_{n-1}); \qquad \varphi_i = \varphi(x_i), \qquad \tau_N = \tau_{N-1}.$$

The system (7)–(12) is nonlinear with respect to the unknowns $\bar{w}_{in}, w_{in}, \tau_n$. Assuming that $\bar{w}_{ik}, w_{ik}, \tau_k,\ 1 \le k \le n-1$, satisfying (7)–(12), are known, in order to determine $\bar{w}_{in}, w_{in}, \tau_n$ $(n = 1, 2, \ldots)$ we apply the method of iterations according to the following scheme:

$$\bar{a}(\bar{w}_{i,n-1})\,\delta_{xx}\bar{w}_{in}^{(s)} - \delta_{\bar{t}}\bar{w}_{in}^{(s)} = 0, \qquad 1 \le i \le n-1; \tag{13}$$

$$\delta_{\bar{t}}\bar{w}_{in}^{(s)} = \frac{1}{\tau_n^{(s)}}(\bar{w}_{in}^{(s)} - \bar{w}_{i,n-1});$$

$$a(w_{i,n-1})\,\delta_{xx} w_{in}^{(s)} - \delta_{\bar{t}} w_{in}^{(s)} = 0, \qquad n+1 \le i \le N-1, \tag{14}$$

$$\delta_{\bar{t}} w_{in}^{(s)} = \frac{1}{\tau_n^{(s)}}(w_{in}^{(s)} - w_{i,n-1});$$

$$\delta_x \bar{w}_{0n}^{(s)} = -\bar{q}_{n-1}; \qquad \delta_x w_{N-1,n}^{(s)} = -q_{n-1}; \qquad w_{i0}^{(s)} = \varphi_i; \qquad \bar{w}_{nn}^{(s)} = w_{nn}^{(s)} = 0; \tag{15}$$

$$\tau_n^{(s+1)} = \frac{1}{\bar{q}_{n-1} - q_{n-1}}\left[\nu h + \tau_n^{(s)}(\bar{q}_{n-1} - q_{n-1} + \delta_x \bar{w}_{n-1,n}^{(s)} - \delta_x w_{nn}^{(s)})\right], \tag{16}$$

where $\tau_n^{(0)} > 0,\ s = 0, 1, 2, \ldots$. Knowing $\tau_n^{(0)} > 0$, from (13)–(15) with $s = 0$ we find $\bar{w}_{in}^{(0)}, w_{in}^{(0)}$, and from (16) with $s = 0$ we then obtain $\tau_n^{(1)}$, etc. In solving the system (13)–(15) one may use the sweep method [7].

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Theorem 1. Let $\bar{q}(t'') \geq \bar{q}(t')$, $q(t'') \leq q(t')$ for any $t'' \geq t' \geq 0$; $\bar{q}(t) > q(t) \geq 0$, $\varphi(0) = 0$; $\delta_x \varphi_i \leq -q(0)$; $\delta_{xx}\varphi_i \geq 0$ for $0 \leq x_i \leq l$; $a(u) \geq a_0 > 0$ for $\varphi(l) \leq u \leq 0$; $\bar{a}(\bar{u}) \geq \bar{a}_0 > 0$ for $l\bar{Q} \geq \bar{u} \geq 0$, $\bar{Q} = \max_{t\geq 0} \bar{q}(t)$. Then, for any prescribed $\tau_n^{(0)} > 0$, the iterations will be uniquely determined for all $s \geq 0$ and, as $s \to \infty$, $\bar{w}_{in}^{(s)}, w_{in}^{(s)}, \tau_n^{(s)}$, changing monotonically, will converge to the solution $\bar{w}_{in}, w_{in}, \tau_n$ of the system (7)–(12).

In this case the following relations hold:

$$-q_{n-1} \geq \delta_x w_{in} \geq -\bar{q}_{n-1}; \qquad \delta_{xx} w_{in} \geq 0; \qquad \delta_t w_{in} \geq 0; \qquad \varphi(l) \leq w_{in} \leq 0; \tag{17}$$

$$0 > \delta_x \bar{w}_{in} \geq -\bar{q}_{n-1}; \qquad \delta_{xx}\bar{w}_{in} > 0; \qquad \delta_t \bar{w}_{in} > 0; \qquad l\bar{Q} \geq \bar{w}_{in} \geq 0; \tag{18}$$

$$\tau_n \geq \frac{\nu h}{\bar{q}_{n-1} - q_{n-1}} > 0.$$

For the proof, note that by means of the maximum principle and induction with respect to n, as in ([5],[6]), one can obtain the estimates:

$$-q_{n-1} \geq \delta_x w_{in}^{(s)}, \qquad \delta_{xx} w_{in}^{(s)} \geq 0; \qquad \delta_t w_{in}^{(s)} \geq 0; \qquad \varphi(l) < w_{in}^{(s)} \leq 0, \tag{17*}$$

$$0 > \delta_x \bar{w}_{in}^{(s)} \geq -\bar{q}_{n-1}; \qquad \delta_{xx}\bar{w}_{in}^{(s)} > 0; \qquad \delta_t \bar{w}_{in}^{(s)} > 0; \qquad l\bar{Q} \geq \bar{w}_{in}^{(s)} \geq 0;$$

$$\tau_n^{(s+1)} \geq \frac{\nu h}{\bar{q}_{n-1} - q_{n-1}} > 0 \tag{18*}$$

for any assignment of $\tau_n^{[0]} > 0$ for all $s \geq 0$. From these inequalities, and also from the maximum principle for the system

$$z_{in}^{(s)} = a(w_{i,n-1})\, \tau_n^{(s)} \delta_{xx} z_{in}^{(s)} + a(w_{i,n-1})\, [\tau_n^{(s)} - \tau_n^{(s-1)}] \delta_{xx} w_{in}^{(s-1)};$$

$$\delta_x z_{N-1,n}^{(s)} = 0; \qquad z_{nn}^{(s)} = 0;$$

$$\bar{z}_{in}^{(s)} = \bar{a}(\bar{w}_{i,n-1})\, \tau_n^{(s)} \delta_{xx} \bar{z}_{in}^{(s)} + \bar{a}(\bar{w}_{i,n-1})\, [\tau_n^{(s)} - \tau_n^{(s-1)}] \delta_{xx} \bar{w}_{in}^{(s-1)};$$

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$$\delta_x \bar{z}^{(s)}_{0n} = 0; \qquad \bar{z}^{(s)}_{nn} = 0;$$

$$\tau_n^{(s+1)} - \tau_n^{(s)} = \frac{h}{q_{n-1} - \bar{q}_{n-1}} \left\{ \sum_{i=1}^{n-1} \frac{z^{(s)}_{in}}{a(w_{i,n-1})} + \sum_{i=n+1}^{N-1} \frac{\bar{z}^{(s)}_{in}}{\bar{a}(\bar{w}_{i,n-1})} \right\},$$

where $z^{(s)}_{in} = w^{(s)}_{in} - w^{(s-1)}_{in}$, $\bar{z}^{(s)}_{in} = \bar{w}^{(s)}_{in} - \bar{w}^{(s-1)}_{in}$, there follows the monotone convergence of $\bar{w}^{(s)}_{in}, w^{(s)}_{in}, \tau^{(s)}_n$ as $s \to \infty$ to the solution $\bar{w}_{in}, w_{in}, \tau_n$ of the system (7)–(12). Here the direction of monotonicity depends on $\tau_n^{(0)} > 0$; in particular, if $0 < \tau_n^{(0)} < \nu h/(\bar{q}_{n-1} - q_{n-1})$, then $\bar{w}^{(s)}_{in}, w^{(s)}_{in}, \tau^{(s)}_n$ increase monotonically with increasing s, while if $\tau_n^{(0)} > 0$ is a sufficiently large number, they decrease monotonically. The estimates (17), (18) follow from (17*), (18*), (12).

3°. The points (y_n, t_n) are joined by straight-line segments, and the polygonal line obtained is denoted by $y(t,h)$, taking $y(t,h) \equiv l$ for $t \geq t_N$. We extend the mesh function w_{in} to the entire domain $D^h_I = \{y(t,h) \leq x \leq l,\ 0 \leq t \leq t_N\}$ discontinuously, as is done in ([8]), p. 359; similarly, $\bar{w}_{in}$ is extended to $D^h_{II} = \{0 \leq x \leq y(t,h),\ 0 \leq t \leq t_N\}$; the resulting functions are denoted respectively by $u(x,t,h)$ and $\bar{u}(x,t,h)$.

Let us call the following set of conditions **A**:

1) $q(t) \equiv q = \text{const} \geq 0$, $\bar{q}(t) \equiv \bar{q} = \text{const}$; $\varphi(x)$ on the interval $0 \leq x \leq l$ is such that $\varphi(0) = 0$, $\varphi'(l) = -q$, $\delta_x \varphi_i \leq -q$, $0 \leq a(\varphi(x_i))\, \delta_{xx}\varphi_i \leq M_2$;

2) $\bar{a}(\bar{u})$, $a(u)$ are defined and have continuous fourth derivatives for $0 \leq \bar{u} \leq l\bar{q}$ and for $\varphi(l) \leq u \leq 0$, respectively; $\bar{a}'(\bar{u}) \leq 0$, $a'(u) \leq 0$, $\bar{a}(\bar{u}) \geq \bar{a}_0 > 0$, $a(u) \geq a_0 > 0$;

3) finally,

$$lM_2 \frac{a_0 + \bar{a}_0}{a_0 \bar{a}_0} + q < \bar{q} < \frac{\nu}{l} \frac{a_0 \bar{a}_0}{a_0 + \bar{a}_0}. \tag{19}$$

If the conditions **A** are fulfilled, then

$$\delta_x w_{in} \geq -\bar{q}, \qquad \delta_t w_{in} \leq \max\{M_2; \bar{q}\Lambda_1\},$$

$$\delta_x \bar{w}_{in} \geq -\bar{q}, \qquad \delta_t \bar{w}_{in} \leq \Lambda_1 \bar{q},$$

where

$$\Lambda_1 = \frac{1}{\nu}(\bar{q} - q).$$

Hence, taking (17), (18) into account,

from (19) and (12) we obtain that

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$$0 < \Lambda_2 \leqslant \frac{y(t'',h) - y(t',h)}{t'' - t'} \leqslant \Lambda_1 \quad \text{for all } t'', t' \geqslant 0, \tag{20}$$

where $\Lambda_2 = \text{const}$, independent of h. In view of inequalities (20), $0 \leqslant y(t,h) \leqslant l$, and Arzelà' s theorem, there exists a sequence h_ν, $\lim_{\nu\to\infty} h_\nu = 0$, such that $y(t, h_\nu)$ converges uniformly for $t \geqslant 0$ to some curve $y(t)$, with

$$\begin{gathered} 0 \leqslant y(t) < l \quad \text{for } 0 \leqslant t < T, \qquad y(T) = l, \qquad y(0) = 0, \\ \Lambda_2 \leqslant \frac{y(t'') - y(t')}{t'' - t'} \leqslant \Lambda_1, \qquad t'', t' \in [0, T]. \end{gathered} \tag{21}$$

Consider the following two auxiliary boundary-value problems: in the domain D_{II} we shall seek a function $\bar{u}(x,t)$ satisfying (1), (3), (5), and in the domain D_I a function $u(x,t)$ satisfying (2), (4), (5), regarding the curve $y(t)$ as prescribed and possessing the properties (21). By the finite-difference method, using inequalities (17), (18), (20) and the estimates of S. N. Bernstein ([8,9]), one can prove that, if conditions A are satisfied, then the functions $\bar{u}(x,t)$, $u(x,t)$ exist, are continuous in the closed domains together with $\bar{u}_x(x,t)$, $u_x(x,t)$, and the derivatives $\bar{u}_{xx}$, $\bar{u}_t$, u_{xx}, u_t are continuous inside the domains of definition of $\bar{u}$ and u and are bounded. We also require the boundedness of $\bar{u}_{xxxx}$, $\bar{u}_{tt}$ in the subdomain $D^\delta_{II}\{0 \leqslant x \leqslant y(t) - \delta,\ t_\delta \leqslant t \leqslant T\}$, $y(t_\delta) = \delta$, and of u_{xxxx}, u_{tt} in $D^\delta_I\{y(t) + \delta \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T_\delta\}$, $y(T_\delta) = l - \delta$, for every δ, where $0 < \delta < l$. We shall call this condition **condition B**.

Theorem 2. *Let conditions A and B be satisfied. Then problem (1)–(6) has a unique solution, and it can be obtained as the limit, as $h \to 0$, of the functions $u(x,t,h)$, $\bar{u}(x,t,h)$, $y(t,h)$ obtained from (7)–(12).*

The proof of this theorem is analogous to the proof of Theorem 2 in ([6]).

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