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Abstract

Full Text

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THEOREMS ON THE TRANSFORMATION AND APPROXIMATION OF ANALYTIC FUNCTIONS

(Presented by Academician S. L. Sobolev on 4 February 1963)

In this article we present without proofs the principal results of an investigation of the so-called analytic transformation $f_g(z)$ of two functions $f(z)$ and $g(t)$, of which the first is analytic and the second periodic. It is established that the transformations $f_{J_n}(z)$, where $J_n(t)$ is the Jackson trigonometric kernel, are algebraic polynomials. With the aid of these polynomials a constructive characterization has been obtained for functions of the classes $W^r H^\alpha$ (r an integer ≥ 0 , $0 < \alpha < 1$) in domains with piecewise smooth boundary, as well as some other results*.

Let K be a bounded nondegenerate continuum with simply connected complement, and let $w = \Phi(z)$ be a function effecting a conformal mapping of the exterior of K onto the exterior of some circle $|w| \leq \rho$, where ρ is chosen so that

$$\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z} = 1.$$

For simplicity we shall assume that $\rho = 1$. Denote by $z = \Psi(w)$ the function inverse to $\Phi(z)$.

In [1] it was established that, in the case when K is the closure of some domain G , an integral of the form

$$\frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} K_n(\sigma) d\sigma \int_C \frac{f(\zeta_{[\sigma]})}{\zeta - z} d\zeta, \tag{1}$$

where $\zeta_{[\sigma]} = \Psi[\Phi(\zeta)e^{-i\sigma}]$, $K_n(\sigma)$ is a trigonometric polynomial of order $\leq n$, and the function $f(\zeta)$ is such that the function it generates, $\hat{f}(t) = f[\Psi(e^{it})]$, is summable on $[-\pi, \pi]$, is, for all $z \in G$, an algebraic polynomial $P_n(f; z)$ of order $\leq n$; moreover, if

$$K_n(\sigma) = \sum_{-n}^n \lambda_k e^{ik\sigma}, \tag{2}$$

$$\hat{f}(t) \sim \sum_{-\infty}^{\infty} c_k e^{ikt}, \tag{3}$$

then

$$P_n(f; z) = \frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} K_n(\sigma) d\sigma \int_C \frac{f(\zeta_{[\sigma]})}{\zeta - z} d\zeta = \sum_0^n \lambda_k c_k \Phi_k(z), \quad (4)$$

where $\Phi_k(z)$ is the k -th Faber polynomial for the domain G .

The following questions arise:

- a) under what conditions on the boundary C and the function $f(\zeta)$, $\zeta \in C$, does there exist

* These results were reported by us on 10 October 1962 at the Second All-Union Conference on Constructive Function Theory in Baku.

the so-called σ -transform $F(z; \sigma)$ of the function $f(\zeta)$:

$$F(z; \sigma)^{\text{def}} = \frac{1}{2\pi i} \int_C \frac{f(\zeta_{[\sigma]})}{\zeta - z} d\zeta, \quad (5)$$

and also the even σ -transform $F(c; z; \sigma)$:

$$F(c; z; \sigma)^{\text{def}} = \frac{1}{2\pi i} \int_C \frac{f(\zeta_{[\sigma]}) + f(\zeta_{[-\sigma]})}{2(\zeta - z)} d\zeta; \quad (6)$$

- b) under what conditions on the periodic function $g(t)$ and on the σ -transform $F(z; \sigma)$ there exists a transform $f_g(z)$ of the functions $f(\xi)$, $\xi \in C$, and $g(t)$:

$$f_g(z)^{\text{def}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) F(z; t) dt = \frac{1}{4\pi i} \int_{-\pi}^{\pi} g(t) dt \int_C \frac{f(\zeta_{[t]})}{\zeta - z} d\zeta; \quad (7)$$

- c) whether a generalization of σ -transforms and of transforms of two functions is possible to the case when the function $f(\xi)$ is given on the boundary C of a bounded continuum K with simply connected complement.

The answers obtained are:

- a) if the boundary C of the domain G is rectifiable and $\hat{f}(t) = f[\Psi(e^{it})]$ is summable and expands in the Fourier series (3), then the σ -transform $F(z; \sigma)$ exists for almost all $\sigma \in [-\pi, \pi]$, and in this case

$$F(z; \sigma) \sim \sum_0^{\infty} c_k \Phi_k(z) e^{-ik\sigma}; \quad (8)$$

- b) if one of the three functions $\hat{f}(t)$, $g(t)$, and $\Psi'(e^{it})$ has bounded variation, and the other two are summable on $[-\pi, \pi]$, then the transform $f_g(z)$ exists, is an analytic function in the domain G , and in this case

$$f_g(z) = \sum_{k=0}^{\infty} c_k d_k \Phi_k(z), \quad (9)$$

where d_k are the Fourier coefficients of the function $g(t)$

$$g(t) \sim \sum_{-\infty}^{\infty} d_k e^{ikt}, \quad (10)$$

- c) for a bounded continuum K with simply connected complement, it is expedient to define the σ -transform by means of the limit (provided it exists)

$$F(z; \sigma) = \lim_{R \rightarrow 1+0} \frac{1}{2\pi i} \int_{C_R} \frac{f \left\{ \Psi \left[\frac{1}{R} \Phi(\zeta) e^{-i\sigma} \right] \right\}}{\zeta - z} d\zeta,$$

where C_R is the level line of the continuum K : $|\Phi(\zeta)| = R$.

In the particular case when the Jackson kernel $J_n(\sigma)$ serves as the kernel $g(\sigma)$, as the transforms $f_{J_n}(\sigma)$ we obtain algebraic polynomials which, depending on the continuum K under consideration, turn into the known trigonometric Jackson polynomials (K is a circle and $z = e^{it}$), into algebraic polynomials by means of which A. F. Timan ⁽²⁾ obtained a strengthening of Jackson's theorem in the periodic case (K is the interval $[-1, 1]$), into algebraic polynomials convenient for approximating analytic functions continuous in domains with rectifiable boundary (see ⁽¹⁾ and Theorems 1-3 of the present paper), and so on.

Theorem 1. *Let G be an arbitrary finite domain with simply connected complement, possessing the property that the function $\Psi(w)$ has, at all points of the circle $|w| = 1$, a nonzero continuous derivative $\Psi'(w)$. Suppose, moreover, that the length of the arc between arbitrary two poi-*

on the boundary C of the domain G does not exceed the distance between these points multiplied by some constant number:

$$\int_{z_1}^{z_2} |d\zeta| \leq A_1 |z_2 - z_1|, \quad z_1, z_2, \zeta \in C; \quad A_1 = \text{const.}$$

Then, for any function $\overline{f}(z)$ analytic in G and having continuous derivatives up to order r inclusive on \overline{G} (r an integer, $r \geq 0$), for every natural n there exists an algebraic polynomial $P_n(z)$ of degree not exceeding n such that, for all $z \in G$,

$$|f(z) - P_n(z)| \leq A_2 \left(\frac{1 + \omega(\Psi'; \frac{1}{n}) \ln n}{n} \right)^r \left\{ \left[1 + \omega(\Psi'; \frac{1}{n}) \ln n \right] \omega\left(f^{(r)}; \frac{1}{n}\right) + \int_0^{1/n} \frac{\omega(f^{(r)}; \sigma) \omega(\Psi'; \sigma)}{\sigma} d\sigma \right\} \quad (11)$$

and, in particular, if for the functions $\Psi'(\omega)$ and $f^{(r)}(z)$, for $|\omega| = 1, z \in C$, the conditions

$$1) \omega\left(\Psi'; \frac{1}{n}\right) = O\left(\frac{1}{\ln n}\right); \quad 2) \int_0^{1/n} \frac{\omega(f^{(r)}; \sigma) \omega(\Psi'; \sigma)}{\sigma} d\sigma = O\left[\omega\left(f^{(r)}; \frac{1}{n}\right)\right],$$

are fulfilled, then for all $z \in G$

$$|f(z) - P_n(z)| \leq A_2' \frac{\omega(f^{(r)}; 1/n)}{n^r}, \quad (11')$$

where the constants A_2 and A_2' depend neither on n nor on the function $f(z)$.

The special feature of this theorem is that the estimates established in it depend both on the properties of the derivative $f^{(r)}(z)$ and on the properties of the boundary C (or, equivalently, on the properties of $\Psi'(\omega)$). This theorem contains, as a special case, a result proved by S. Ya. Al' per ([3], Theorem 4) on the approximation of functions analytic in G and continuous on \bar{G} .

Theorem 2. Let G be a finite domain with a simply connected complement, whose boundary C consists of a finite number of arcs $C^{(j)}$ ($j = 1, 2, \dots, k$) with continuous curvature, forming at their junction points z_j angles $\alpha_j \pi, 0 < \alpha_j < 1$, and having the property that, in a neighborhood of each of the points $\omega_j = \Phi(z_j)$, the function $z = \Psi(\tau)$ can be represented in the form

$$\Psi(\tau) = \lambda(\tau)(\tau - \omega_j)^{2-\alpha_j} + \Psi(\omega_j),$$

where $\lambda(\tau)$ is a function continuous along $|\tau| = 1$ in some neighborhood of the point ω_j , together with its first and second derivatives $\lambda'(\tau), \lambda''(\tau)$, and $\lambda(\omega_j) \neq 0$.

Then, if on \bar{G} a function $f(z)$ is given, analytic in G and continuous on \bar{G} , with modulus of continuity $\omega(t) = \omega(f; t)$, then for $f(z)$, for every natural n , one can construct an algebraic polynomial $P_n(z) = P_n(f; z)$ of degree not exceeding n such that at all points $z \in C$ the inequality

$$|f(z) - P_n(z)| \leq A_3 \omega \left[\rho_{1+1/n}(z) \right]$$

will hold, where $\rho_{1+1/n}(z)$ is the distance from the point z to the level line $|\Phi(z)| = 1 + \frac{1}{n}$, and A_3 is a constant independent of both z and n .

Theorem 3. If, under the conditions of the preceding theorem, in the domain G a function $f(z)$, being analytic in G , has on the closed domain \overline{G} continuous derivatives up to order r inclusive (r an integer, $r \geq 0$) and, moreover, for some $a \in (0, 1]$ the derivative $f^{(r)}(z)$ satisfies a Hölder condition of order a , then for the function $f(z)$, for every natural n , one can

to construct an algebraic polynomial $P_n(z)$ of degree not exceeding n such that at all points $z \in C$ the inequality

$$|f(z) - P_n(z)| \leq A_4[\rho_{1+1/n}(z)]^{r+\alpha},$$

holds, where the constant A_4 depends neither on $z \in C$ nor on n .

From this theorem and Theorems 4.1 and 2.4 from ⁽⁴⁾ there follows the following theorem, giving a constructive characterization in domains with corners for analytic functions $f(z)$, whose r -th derivative belongs to the class H^α , $0 < \alpha < 1$.

Theorem 4. In order that a function $f(z)$, analytic in a domain G with corners satisfying the conditions of Theorem 2, have in the closed domain a continuous derivative of order r (r arbitrary integer), $f^{(r)}(z)$, belonging to the Hölder class H^α , $0 < \alpha < 1$, it is necessary and sufficient that for this function, for each $n = 1, 2, \dots$, there exist an algebraic polynomial $P_n(z)$ of degree not exceeding n such that, for all z belonging to the boundary C of the domain G , the inequality

$$|f(z) - P_n(z)| \leq A_5[\rho_{1+1/n}(z)]^{r+\alpha}$$

holds, where the constant A_5 depends neither on z nor on n , and $\rho_{1+1/n}(z)$ denotes the distance from the point $z \in C$ to the level line

$$|\Phi(z)| = 1 + \frac{1}{n}.$$

In addition, we have considered questions relating to the approximation of analytic functions given in domains with corners $> \pi$, on the simplest Riemann surfaces, and on a finite number of continua.

Theorems analogous to Theorem 2 also hold for the approximation of logarithmic potentials by harmonic polynomials.

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CITED LITERATURE

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² A. F. Timan, *DAN*, **78**, 17 (1951).

³ S. Ya. Al' per, *Izv. AN SSSR, ser. matem.*, **19**, 423 (1955).

⁴ V. K. Dzyadyk, *Izv. AN SSSR, ser. matem.*, **23**, 697 (1959).

Note: Figure translations are in progress. See original paper for figures.

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