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Abstract

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S. ILIADIS

CHARACTERIZATION OF SPACES BY MEANS OF H-CLOSED EXTENSIONS*

(Presented by Academician P. S. Aleksandrov on 29 X 1962)

In the paper (1) Katětov posed several questions concerning the maximal H-closed extension τR.

Here are two of them:

1. Under what conditions imposed on the space R do the extension τR and the Čech extension βR coincide? The answer to this question is given by Theorem 12 of (3).
2. It is known that if completely regular spaces R_1 and R_2 satisfy the first axiom of countability, then from a homeomorphism of the extensions βR_1 and βR_2 there follows a homeomorphism of the spaces R_1 and R_2.

It would be interesting to find sufficiently broad conditions for analogous assertions concerning the extension τR.

Here a new proof is given of Theorem 12 of (3). In Theorem 5 there are given, as it seems to me, the conditions required in the second question. In conclusion, some properties are considered of points of a special type, called here "maximal."

§ 1. A point x of the space R will be called a **point of contact** of a set M, lying in R, if $x \notin M$, $x \in \overline{M}$** and $x \in R \setminus \overline{M}$. If some non-isolated point ξ is not a point of contact for any set, then we shall call it a **maximal point**.

Theorem 1. *If $\xi \in \tau R \setminus R$, then ξ is maximal in τR.*

Proof. First let us recall how the space τR is constructed (see (1)). A system γ, consisting of nonempty open sets $\Gamma \subset R$, will be called an α-system if it satisfies the following two conditions: 1) from the inclusions $\Gamma_1 \in \gamma$ and $\Gamma_2 \in \gamma$ there follows the inclusion $\Gamma_1 \cap \Gamma_2 \supset \gamma$; 2)

$$\bigcap_{\Gamma \in \gamma} \overline{\Gamma} = \emptyset^{***}.$$

As usual, an α-system will be called **maximal** if every α-system containing it coincides with it.

The points of the space τR are all points of the space R and all maximal α-systems ξ. A base of the space τR at a point $x \in R$ is formed by the system of neighborhoods of the point x in R. A base of the space τR at a point ξ is formed by the sets $\xi \cup \Gamma$, where $\Gamma \in \xi$. It is not hard to show (see ([1])) that τR is the maximal H-closed extension of the space R. From the construction it is seen that R is open in τR, and $\tau R \setminus R$ is discrete.

We now prove the theorem. Let $\xi \in \tau R \setminus R$. Suppose that there exists a set M such that $\xi \notin M$, $\xi \in \overline{M}$, and $\xi \in \tau R \setminus \overline{M}$ (here $\overline{M}$ is the closure in τR). The set $\tau R \setminus \overline{M}$ is open in τR. From the maximality of the system ξ it follows that

$$R \cap (\tau R \setminus \overline{M}) \in \xi,$$

since $R \cap (\tau R \setminus \overline{M})$ intersects every neighborhood of the point ξ. But this contradicts the condition $\xi \in \overline{M}$. The theorem is proved.

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* All spaces considered are Hausdorff. An extension tR of a space R is called H-closed if it is closed in every (topological) space containing it.

** $\overline{M}$ is the closure of the set M in R.

*** $\emptyset$ is the empty set.

Definition 1 (see (2)). An infinite set E of the space R **converges in cardinality** to a point x of the space R, if for every neighborhood Ox we have the inequality

$$\operatorname{card}(E \cap Ox) > \operatorname{card}(E \setminus Ox).$$

Theorem 2. *No set E of the space R of regular cardinality can converge in cardinality to any maximal point ξ of the space R.*

Proof. Let the set E be of regular cardinality $\aleph_\tau$ and converge in cardinality to the maximal point ξ. Write all points of the set E in the form of a sequence of type ω_τ:

$$x_1, x_2, \dots, x_\lambda, \dots; \qquad \lambda < \omega_\tau.$$

By Φ_λ denote the closure of the set of all points $x_{\lambda'}$ for which $\lambda' \geq \lambda$. We first prove that $\xi \in \langle \Phi_\lambda \rangle^*$ for every λ.

Indeed, if $\xi \notin \langle \Phi_\lambda \rangle$, then $\xi \in R \setminus \Phi_\lambda$. Let, further, $M = \Phi_\lambda \setminus \xi$. It is clear that $\xi \notin M$, $\xi \in \overline{M} = \Phi_\lambda$. All this contradicts the fact that ξ is a maximal point. Hence $\xi \in \langle \Phi_\lambda \rangle$.

To complete the proof we shall need the following

Lemma 1. *For every index λ there is an index α such that*

$$\langle \Phi_\lambda \rangle \setminus \Phi_\alpha \neq \emptyset.$$

Proof. Let $x \neq \xi$ and $x \in \langle \Phi_\lambda \rangle$. Take disjoint neighborhoods $O\xi$ and Ox of the points ξ and x.

Recall that $\operatorname{card}(E \setminus O\xi) < \operatorname{card} E$; consequently, by regularity of the number ω_τ, there is an α such that $\lambda' < \alpha$ as soon as $x_{\lambda'} \in R \setminus O\xi$. Therefore

$$\Phi_\alpha \subseteq \overline{O\xi} \subseteq R \setminus Ox,$$

and hence $x \in \langle \Phi_\lambda \rangle \setminus \Phi_\alpha$. The lemma is proved.

We continue the proof of the theorem. Take a sequence

$$\Phi_{\lambda_1}, \Phi_{\lambda_2}, \dots, \Phi_{\lambda_\alpha}, \dots,$$

such that

$$\langle \Phi_{\lambda_\alpha} \rangle \setminus \Phi_{\lambda_{\alpha+1}} \neq \emptyset$$

for every α, and such that the sequence of all indices λ_α is cofinal with the sequence

$$0, 1, 2, \dots, \lambda, \dots, \qquad \lambda < \omega_\tau.$$

Such a sequence exists, as follows from the lemma. Let

$$H_\alpha = \langle \Phi_{\lambda_\alpha} \rangle \setminus \Phi_{\lambda_{\alpha+1}}.$$

If $\alpha' \neq \alpha''$, then $H_{\alpha'} \cap H_{\alpha''} = \emptyset$. In the system $\eta = \{H_\alpha\}$, consisting of all sets H_α, consider the subsystem η_0 consisting of all those sets H_α for which $\alpha = \alpha' + n$, where n is an even natural number, and α' is a limit transfinite number or 0. Let

$$U = \bigcup_{H_\alpha \in \eta_0} H_\alpha \qquad \text{and} \qquad V = \bigcup_{H_\alpha \notin \eta_0} H_\alpha.$$

The sets U and V are nonempty, open, and disjoint. It is clear that $\xi \notin U$ and $\xi \notin V$. We shall show that

$$O\xi \cap (\langle \Phi_{\lambda_\alpha} \rangle \setminus \Phi_{\lambda_{\alpha+1}}) \neq \emptyset$$

for every neighborhood $O\xi$ of the point ξ and for all indices λ_α, starting with some index λ (where λ depends on $O\xi$). Let $O\xi$ be a neighborhood of the point ξ. There exists a λ such that

$$\Phi_\lambda \subseteq \overline{O\xi}.$$

Let $\lambda_\alpha > \lambda$. It is clear that

$$\Phi_{\lambda_\alpha} \subseteq \Phi_\lambda \subseteq \overline{O\xi}.$$

Further, the boundary $\overline{O\xi} \setminus O\xi$ contains no nonempty open set, while the set

$$\langle \Phi_{\lambda_\alpha} \rangle \setminus \Phi_{\lambda_{\alpha+1}}$$

is nonempty, open, and is contained in $O\xi$. But the set

$$\langle \Phi_{\lambda_\alpha} \rangle \setminus \Phi_{\lambda_{\alpha+1}} = H_\alpha$$

either belongs to η_0 or does not belong to η_0, depending on the "parity" of the number α. Hence $\xi \in \overline{U}$ and $\xi \in \overline{V}$. Since $V \subseteq R \setminus \overline{U}$, it follows that $\xi \in R \setminus \overline{U}$, which is impossible. Theorem 2 is completely proved.

Definition 2 (see (2)). A point x will be called an x-point of the space R, if there exists a countable set E converging to the point x.

Corollary. *A maximal point ξ of the space R cannot be an x-point.*

We shall now need Theorem 3 and its corollary, due to P. S. Aleksandrov (see (2), Ch. IV, § 1, item 7).

* $\langle M \rangle$ is the open kernel of the set M in the space R.

Theorem 3. If at a non-isolated point x the space R is regular and if $\psi_x R = \chi_x R^*$, then there exists a set E of regular cardinality which converges in cardinality to the point $x \in R$.

Corollary. To every point x of a bicompactum R there converges in cardinality some set E_x of regular cardinality.

From this corollary and from Theorems 1 and 2 it follows immediately:

Theorem 4 (first main theorem). If R is a completely regular non-bicompact space, then τR and βR do not coincide.

Theorem 5 (second main theorem). Suppose the spaces R_1 and R_2 contain no maximal points. If the extensions τR_1 and τR_2 are homeomorphic, then the spaces R_1 and R_2 are also homeomorphic.

Proof. Let $\tau R = \tau R_1 = \tau R_2$. The spaces R_1 and R_2 may be regarded as subsets of the space τR. We shall need the following

Lemma 2. If the set $R_1 \cup \xi$ is everywhere dense and open in the space R, and the point ξ is maximal in R, then it is maximal also in $R_1 \cup \xi$.

Proof. Suppose that the point ξ is maximal in R, but not maximal in $R_1 \cup \xi$. Then there is a subset M of the set $R_1 \cup \xi$ such that $\xi \notin M$, $\xi \in \overline{M}$, and

$$\xi \in (R_1 \cup \xi) \setminus \overline{M}.$$

Here closures are taken in the space $R_1 \cup \xi$. The set

$$U = (R \cup \xi) \setminus \overline{M}$$

is open in R, and $\xi \in \overline{U}^R$. Let $N = R \setminus (U \cup \xi)$. It is clear that $\xi \notin N$, $\xi \in \overline{N}^R$, but

$$\xi \in R \setminus \overline{N}^R,$$

since $U \subset R \setminus \overline{N}^R$. This contradicts our assumption. The lemma is proved.

We return to the proof of the theorem. Suppose that $R_1 \setminus R_2 \neq \emptyset$, and let $\xi \in R_1 \setminus R_2$. By Theorem 1 the point ξ is maximal in τR, and by Lemma 2 it is maximal in the space $R_1 \cup \xi$, contrary to the assumption. Hence $R_1 \subseteq R_2$. In the same way one proves that $R_2 \subseteq R_1$. Thus $R_1 = R_2$. Theorem 5 is proved.

Corollary. Suppose all points of the spaces R_1 and R_2 are χ-points; if the extensions τR_1 and τR_2 are homeomorphic, then the spaces R_1 and R_2 are also homeomorphic.

§ 2. By $\psi'_\xi R$ we denote the least of all cardinal numbers ψ such that the point ξ has in the space R a system of neighborhoods $\{O_\alpha \xi\}$ of cardinality ψ, possessing the property

$$\bigcap_\alpha \overline{O_\alpha \xi} = \xi.$$

Theorem 6. If ξ is a maximal point of an H-closed space R, then

$$\psi_\xi R \neq \psi'_\xi R.$$

Proof. Let $\psi_\xi R = \psi'_\xi R$, and let $\{O_\alpha \xi\}$ be such a system of neighborhoods $O_\alpha \xi$ of the point ξ, of cardinality $\psi'_\xi R = \aleph_\tau$, that

$$\bigcap \overline{O_\alpha \xi} = \xi.$$

Write the elements of this system in a sequence of type ω_τ:

$$O_1\xi,\ O_2\xi, \dots,\ O_\alpha\xi, \dots; \qquad \alpha < \omega_\tau.$$

Put $H_1 = O_1\xi$. Suppose that neighborhoods H_β of the point ξ have been constructed for all $\beta < \alpha$, possessing the property that, if $\beta_1 \geq \beta_2$, then

$$\overline{H_{\beta_1}} \subseteq \overline{H_{\beta_2}}.$$

We shall prove that

$$\bigcap_{\beta<\alpha<\omega_\tau} \overline{H_\beta}$$

contains some neighborhood of the point ξ. The point ξ cannot be isolated in the intersection $\bigcap_{\beta<\alpha} \overline{H_\beta}$, for otherwise the cardinality of the system $\{O_\alpha \xi\}$ would not be equal to $\psi'_\xi R$.

Let

$$M = \left(\bigcap_{\beta<\alpha} \overline{H_\beta}\right) \setminus \xi.$$

Since ξ is not isolated in $\bigcap_{\beta<\alpha} \overline{H_\beta}$, we have

$$M = \bigcap_{\beta<\alpha} \overline{H_\beta}.$$

But if $\bigcap_{\beta<\alpha} \overline{H_\beta}$ contains no neighborhood of the point ξ, then

$$\xi \in R \setminus \left(\bigcap_{\beta<\alpha} \overline{H_\beta} \right) = R \setminus \overline{M},$$

which is impossible, since the point ξ is maximal in R. Hence

$$\bigcap_{\beta<\alpha} \overline{H_\beta}$$

contains a neighborhood of the point ξ. By H_α we shall denote such a neighborhood of the point ξ, which is contained in $O_\alpha \xi \cap \left(\bigcap_{\beta<\alpha} \overline{H}_\beta \right)$. It is clear that $\overline{H}_\alpha \subseteq \overline{H}_\beta$, if $\alpha \geq \beta$, and that $\bigcap_\alpha \overline{H}_\alpha = \xi$. For every α the set $\langle \overline{H}_\alpha \rangle$ is nonempty. We may assume that $\langle \overline{H}_\alpha \rangle \setminus \overline{H}_\beta$ is nonempty if $\beta > \alpha$, for otherwise we could pass to a subsequence having this property. Let ξ_α be some point belonging to the set $\langle \overline{H}_\alpha \rangle \setminus \overline{H}_{\alpha+1}$. Since $\langle \overline{H}_\beta \rangle \supseteq \langle \overline{H}_\alpha \rangle$, if $\alpha \geq \beta$, the set of points $\xi_{\alpha'}$ belonging to the set $R \setminus \langle \overline{H}_\alpha \rangle$ has cardinality less than $\psi_\xi R = \aleph_\tau$. Let $x \in R$. Since $\bigcap_\alpha \overline{H}_\alpha = \xi$, we have $x \notin \overline{H}_{\alpha_0}$ for some α_0. Hence $Ox = R \setminus \overline{H}_{\alpha_0}$ is a neighborhood of the point x. But $\overline{Ox} = R_1 \setminus \overline{H}_{\alpha_0}$ is contained in $R \setminus \langle \overline{H}_{\alpha_0} \rangle$, hence

$$\operatorname{card}(E \cap \overline{Ox}) < \operatorname{card} E,$$

where E is the set of all points ξ_α. Hence, by virtue of the H-closedness of the space R, it follows that $\operatorname{card}(\overline{O\xi} \cap E) = \aleph_\tau$ and $\operatorname{card}(E \setminus O\xi) < \aleph_\tau$, where $O\xi$ is any neighborhood of ξ. Let $G_\alpha = \langle \overline{H}_\alpha \rangle \setminus \overline{H}_{\alpha+1}$. It is clear that $\xi_\alpha \in G_\alpha$. We divide the system $\gamma = \{G_\alpha\}$ into two subsystems γ_1 and γ_2. We put $G_\alpha \in \gamma_1$ if $\alpha = \alpha' + n$, where n is an even natural number, and α' is a limit transfinite number or 0. We set $U = \bigcup_{G_\alpha \in \gamma_1} G_\alpha$. If $G_\alpha \notin \gamma_1$, then $G_\alpha \in \gamma_2$. Let $V = \bigcup_{G_\alpha \in \gamma_2} G_\alpha$. We shall show that $O\xi \cap G_\alpha \neq \emptyset$ for any neighborhood $O\xi$ of the point ξ and for all indices α, beginning with some index α_0 (α_0 depends on $O\xi$). This follows from the fact that $\overline{O\xi}$ contains all points except, possibly, some set of cardinality less than $\aleph_\tau$. Hence it follows that $\xi \in \overline{U}$ and $\xi \in \overline{V}$. But $\overline{V} \subseteq R \setminus \overline{U}$, which is impossible by the maximality of the point ξ. The theorem is proved.

Corollary 1. If an H-closed space R is regular at its point ξ, then this point is not maximal.

Corollary 2. If ξ is a maximal point of an H-closed space R, then $\psi_\xi R \neq \chi_\xi R$.

It is not difficult to show that $\psi_x \tau R = \psi_x R$ and $\chi_x \tau R = \chi_x R$ for all points $x \in R$.

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[*] χ_{xR} (respectively ψ_{xR}) is the least cardinal number which is the cardinality of some base (respectively pseudobase) of the space R at the point x.

Corollary 3. If $\psi_x R_1 = \chi_x R_1$ at all points of the space R_1 and $\psi_x R_2 = \chi_x R_2$ at all points of the space R_2, then from the homeomorphism of the extensions τR_1 and τR_2 there follows a homeomorphism of the spaces R_1 and R_2.

Remark 1. In Corollary 1 one cannot dispense with H-closedness. Indeed, let the space R consist of isolated points and let $\xi \in \tau R \setminus R$. Then the point ξ in the space $R \cup \xi$ is maximal, while the space $R \cup \xi$ is regular at it.

Remark 2. It is not difficult to prove that if the point ξ is maximal in R, then it is also maximal in τR. Since pseudocharacter and character are preserved upon passing to the extension τR, Corollary 2 is true not only for H-closed spaces, but also for any space R.

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Moscow State University
named after M. V. Lomonosov

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CITED LITERATURE

1. M. Katětov, Čas. matem. fysiky, **72**, 17 (1947).
2. P. S. Aleksandrov, P. S. Uryson, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **31** (1950).
3. M. Katětov, Čas. matem. fysiky, **72**, 101 (1947).

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