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Abstract

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ON A BOUNDARY-VALUE PROBLEM ON THE AXIS FOR A NONLINEAR EQUATION OF THE n-TH ORDER

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1. We shall be interested in those solutions $x(t)$ of the autonomous nonlinear equation

$$x^{(n)} + f\left(x, \dot{x}, \dots, x^{(n-1)}\right) = 0, \qquad (1)$$

which satisfy the following boundary conditions:

$$x(-\infty) = q_1 \ (q_1 < 0), \qquad x(+\infty) = q_2 \ (q_2 > 0),$$

$$x^{(s)}(\pm\infty) = 0 \quad (s = 1, 2, \dots, n).$$

We shall call such solutions **transition** solutions.

As regards the function f, we shall assume that it is continuous in the domain $q_1 \le x \le 0,\ 0 \le x \le q_2,\ -\infty < x^{(s)} < \infty$ $(s = 1, 2, \dots, n-1)$, and satisfies the relations

$$f(q_i, 0, \dots, 0) = 0 \ (i = 1, 2), \qquad f(x, 0, \dots, 0) \ne 0 \ (q_1 < x < q_2). \qquad (2)$$

Yu. A. Klokov [1] considered an analogous problem for the case $n = 3$, but under the opposite assumption $f(x, 0, 0) \equiv 0$.

Below we give two sufficient conditions for the existence of transition solutions of equation (1) and note some properties of these solutions.

2. We introduce a number of auxiliary notations which will help formulate the results obtained. Consider a pair of polynomials

$$P_{\pm}(\nu) = a_n \nu^n + a_{n-1} \nu^{n-1} + \dots + a_1 \nu + a_{\pm},$$

possessing the following two properties: 1) they differ only in their constant terms, with $a_- > 0$, $a_+ < 0$, and $a_1 < 0$; 2) the polynomial $P_-(\nu)$ has l negative zeros $\lambda_1^- > \lambda_2^- > \dots > \lambda_l^-$ and m positive zeros $\mu_1^- < \mu_2^- < \dots < \mu_m^-$ $(l+m=n)$; the polynomial $P_+(\nu)$ has $(l+1)$ negative zeros $\lambda_1^+ > \lambda_2^+ > \dots > \lambda_{l+1}^+$ and $(m-1)$ positive zeros $\mu_1^+ < \mu_2^+ < \dots < \mu_{m-1}^+$. We shall denote the totality of all such pairs of polynomials by Π.

Denote by $K_-(P_\pm)$ the cone consisting of all polynomials $Q(\nu)$ of the form

$$\begin{aligned} Q(\nu) = c_0 &+ c_1\left(1-\frac{\nu}{\mu_1^+}\right) + c_2\left(1-\frac{\nu}{\mu_1^+}\right)\left(1-\frac{\nu}{\mu_2^+}\right) + \cdots \\ &+ \cdots + c_{m-1}\left(1-\frac{\nu}{\mu_1^+}\right)\left(1-\frac{\nu}{\mu_2^+}\right)\cdots\left(1-\frac{\nu}{\mu_{m-1}^+}\right) \\ &+ \prod_1^m\left(1-\frac{\nu}{\mu_j^-}\right)\left\{c_m + c_{m+1}\left(1-\frac{\nu}{\lambda_1^-}\right) + c_{m+2}\left(1-\frac{\nu}{\lambda_1^-}\right)\left(1-\frac{\nu}{\lambda_2^-}\right)\right. \\ &\left.\qquad + \cdots + c_{n-1}\left(1-\frac{\nu}{\lambda_1^-}\right)\cdots\left(1-\frac{\nu}{\lambda_{l-1}^-}\right)\right\}, \end{aligned}$$

where all coefficients c_j $(j = 0, 1, \dots, n-1)$ are nonnegative.

In a similar way, denote by $K_+(P_\pm)$ the cone consisting of all polynomials $Q(\nu)$ of the form

$$Q(\nu) = c_0 + c_1\left(1-\frac{\nu}{\lambda_1^-}\right) + c_2\left(1-\frac{\nu}{\lambda_1^-}\right)\left(1-\frac{\nu}{\lambda_2^-}\right) + \cdots$$

$$\begin{aligned} &\cdots + c_l\left(1-\frac{\nu}{\lambda_1^-}\right)\left(1-\frac{\nu}{\lambda_2^-}\right)\cdots\left(1-\frac{\nu}{\lambda_l^-}\right) + \prod_1^{l+1}\left(1-\frac{\nu}{\lambda_j^+}\right)\times \\ &\times\left\{c_{l+1} + c_{l+2}\left(1-\frac{\nu}{\mu_1^+}\right) + \cdots + c_{n-1}\left(1-\frac{\nu}{\mu_1^+}\right)\left(1-\frac{\nu}{\mu_2^+}\right)\cdots\left(1-\frac{\nu}{\mu_{m-2}^+}\right)\right\}, \end{aligned}$$

where all coefficients c_j $(j = 0, 1, \dots, n-1)$ are nonnegative.

The question of whether an arbitrary polynomial belongs to the cone $K_-(P_\pm)$ or $K_+(P_\pm)$ is decided in an elementary way.

Let H_n be the n-dimensional space of vectors $\{f_i\}_0^{n-1}$. In it we single out the set $F_-(P_\pm)$ of vectors distinguished by the following property:

$$P_-(v) - a_n(v^n + f_{n-1}v^{n-1} + \cdots + f_1 v + f_0) \in K_-(P_\pm).$$

In the same way, by means of the condition

$$P_+(v) - a_n(v^n + f_{n+1}v^{n-1} + \dots + f_1 v + f_0) \in K_+(P_\pm)$$

we single out the set $F_+(P_\pm)$.

Let us introduce the differential operators

$$M_j^- = -\frac{1}{\mu_j^-}\frac{d}{dt} + 1 \quad (j = 1, 2, \dots, m);$$

$$M_j^+ = -\frac{1}{\mu_j^+}\frac{d}{dt} + 1 \quad (j = 1, 2, \dots, m-1);$$

$$L_j^- = -\frac{1}{\lambda_j^-}\frac{d}{dt} + 1 \quad (j = 1, 2, \dots, l);$$

$$L_j^+ = -\frac{1}{\lambda_j^+}\frac{d}{dt} + 1 \quad (j = 1, 2, \dots, l+1).$$

With the aid of these operators we associate with the function $x(t)$ n functions $y_j^-(t)$ $(j = 0, 1, \dots, m-1)$, $z_k^-(t)$ $(k = 0, 1, \dots, l-1)$, defined on the negative half-axis in the following way:

$$y_0^-(t) = x(t); \quad z_0^-(t) = M_1^- M_2^- \cdots M_m^- x(t); \quad y_j^-(t) = M_j^+ y_{j-1}^-(t); \tag{3}$$

$$z_k^-(t) = L_k^- z_{k-1}^-(t) \quad (t < 0;\ j = 1, 2, \dots, m-1;\ k = 1, 2, \dots, l-1).$$

On the positive half-axis we define the functions $y_j^+(t)$ $(j = 0, 1, \dots, l)$ and $z_k^+(t)$ $(k = 0, 1, \dots, m-2)$

$$y_0^+(t) = x(t); \quad z_0^+(t) = L_1^+ L_2^+ \cdots L_{l+1}^+ x(t); \quad y_j^+(t) = L_j^- y_{j-1}^+(t); \tag{4}$$

$$z_k^+(t) = M_k^+ z_{k-1}^+(t) \quad (t > 0;\ j = 1, 2, \dots, l;\ k = 1, 2, \dots, m-2).$$

Denote by D_- (D_+) that domain of values in the space of the variables $x, \dot{x}, \dots, x^{(n-1)}$ which corresponds to the cube $q_1 \le y_j^-, z_k^- \le 0$ $(0 \le y_j^+, z_k^+ \le q_2)$ in the space of the variables y^-, z^- (y^+, z^+).

We shall say that the function $f(x, \dot{x}, \dots, x^{(n-1)})$ is **majorized** by the pair $P_\pm \in \Pi$ if: 1) the function $f(x, \dot{x}, \dots, x^{(n-1)})$ is differentiable at every point of the domains D_- and D_+; 2) the function f satisfies condition (2), and the sign of the function $f(x, 0, \dots, 0)$ in the interval $q_1 < x < q_2$ coincides with the sign of the coefficient a_n; 3) $a_- q_1 = a_+ q_2$; 4) the gradient of the function f (i.e. the vector with coordinates

$$f_j = \frac{\partial f(x, \dot{x}, \dots, x^{(n-1)})}{\partial x^{(j)}})$$

at every point of the domain D_- belongs to the set $F_-(P_\pm)$, and at every point of the domain D_+, to the set $F_+(P_\pm)$.

3. Let us formulate the results obtained.

Theorem 1. *If there exists a pair of polynomials $P_\pm \in \Pi$ which majorizes the function $f(x, \dot{x}, \dots, x^{(n-1)})$, then equation (1) has a transitional solution $x(t)$ satisfying the conditions*

$$q_1 \le x(t) \le 0 \quad (t \le 0), \qquad 0 \le x(t) \le q_2 \quad (t \ge 0). \tag{5}$$

Theorem 1 can be made somewhat more precise. To this end, denote by S the totality of all n-times differentiable functions $x(t)$ possessing

with the following property: for every $t < 0$ $(t > 0)$ the set of n numbers $\{x(t), \dot{x}(t), \dots, x^{(n-1)}(t)\}$ belongs to the domain D_- (D_+). If $x_1(t) - x_2(t) \in S$, then we shall say that the function $x_1(t)$ is steeper than the function $x_2(t)$, and write $x_1 \gg x_2$.

Theorem 2. *Suppose that all the conditions of Theorem 1 are fulfilled. Then equation (1) has a solution $x(t) \in S$. If there are several such solutions, then among them there exist two "extreme" solutions: the steepest $x_1(t)$ and the least steep $x_2(t)$.*

The following theorem shows how transition solutions change when the function f is varied.

Theorem 3. *Let two functions $f_i(x, \dot{x}, \dots, x^{(n-1)})$ $(i = 1, 2)$ be majorized by one and the same pair of polynomials $P_\pm \in \Pi$. Suppose that in the domain $D_- + D_+$ the inequality $a_n f_1 \gg a_n f_2$ holds. Then the extreme solutions x_1 and x_2 of equation (1) for $f = f_1$ are steeper than those for $f = f_2$.*

Theorem 1 can be considerably strengthened if one uses Schauder' s fixed-point theorem ([2]). This gives

Theorem 4. *Let $f_i(x, \dot{x}, \dots, x^{(n-1)})$ $(i = 1, 2)$ be any two functions majorized by some pair $P_\pm \in \Pi$. If the function $f(x, \dot{x}, \dots, x^{(n-1)})$ is continuous in the domains D_- and D_+, satisfies condition (2), and if in the domain $D_- + D_+$ the inequality $a_n f_1 \ll a_n f \ll a_n f_2$ holds, then equation (1) has at least one solution $x(t) \in S$. Moreover, if $x_i(t)$ is some transition solution of equation (1) for $f = f_i$ $(i = 1, 2)$, and the function $x_1(t)$ is less steep than $x_2(t)$, then among the functions $x(t)$ such that $x_1 \ll x \ll x_2$ there exists at least one transition solution of equation (1).*

4. Let us outline the proof of the theorems of the preceding section. Equation (1) is evidently equivalent to the equation

$$P\left(\frac{d}{dt}\right)x + a(x)x =$$

$$= -a_n f(x, \dot{x}, \dots, x^{(n-1)}) + a_{n-1}x^{(n-1)} + \dots + a_1\dot{x} + a(x)x \equiv$$

$$\equiv g(x, \dot{x}, \dots, x^{(n-1)}), \tag{6}$$

where $P(\nu) = a_n\nu^n + a_{n-1}\nu^{n-1} + \dots + a_1\nu$; $a(x) = a_-$, if $x < 0$, and $a(x) = a_+$, if $x > 0$. We shall seek solutions of this equation among functions $x(t)$ satisfying condition (5). For such functions the relation $a(x(t)) = a(t)$ is valid. Therefore the nonlinear operator $P(d/dt) + a(x)$, standing on the left-hand side of (6), can be replaced by the linear operator $L = P(d/dt) + a(t)$. The operator L has a Green' s function $K_0(t, s)$ satisfying the conditions $K_0(\pm\infty, s) = K_0(0, s) = 0$. With its aid, equation (6) can be replaced by the equivalent integro-differential equation

$$x(t) = \int_{-\infty}^{\infty} K_0(t, s)\, g(x(s), \dot{x}(s), \dots, x^{(n-1)}(s))\, ds. \tag{7}$$

Using definitions (3) and (4), it is easy to obtain from this

$$y_j^-(t) = \int_{-\infty}^{\infty} K_j^-(t, s)\, g(x(s), \dot{x}(s), \dots, x^{(n-1)}(s))\, ds$$

$$(t < 0;\ j = 0, 1, 2, \dots, m-1),$$

where $K_j^-(t, s) = M_j^+ M_{j-1}^+ \cdots M_1^+ K_0(t, s)$ $(t < 0)$. Analogous relations are also obtained for the functions $z_k^-(t)$ for $t < 0$ and the functions $y_j^+(t)$ and $z_k^+(t)$ for $t > 0$. If in these relations the functions $x(s), \dot{x}(s), \dots, x^{(n-1)}(s)$ are replaced by the corresponding linear combinations of the functions $y_j^-(s)$

and $z_k^-(s)$ for $s < 0$ and by combinations of the functions $y_j^+(s)$ and $z_k^+(s)$ for $s > 0$, then one obtains a system of integral equations which is equivalent on S to equation (1). Let $K = K_j^-(t, s)$ $(K = K_j^+(t, s))$ be the kernel of one of the equations of this system. For $t < 0$ $(t > 0)$ it has the following properties: a) $a(t)K(t, s) > 0$; b)

$$0 \le a(t) \int_{-\infty}^{\infty} K(t, s)\, ds \le 1;$$

c) the function

$$\int_{-\infty}^{\infty} K(t,s)\,ds$$

decreases monotonically as t increases. The validity of these assertions is easily verified if one uses the following representation of the Green' s function $K_0(t,s)$ ([3]):

$$K_0(t,s) = \frac{a_- - a_+}{a_n^2} \int_0^t Y(\lambda^+, \mu^-, t-\sigma)\, Y(\lambda^-, \mu^+, \sigma - s)\, d\sigma, \tag{8}$$

where

$$Y(\lambda^+, \mu^-, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{\nu t}\, d\nu}{\Pi(\nu - \lambda^+)\Pi(\nu - \mu^-)};$$

$$Y(\lambda^-, \mu^+, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{\nu t}\, d\nu}{\Pi(\nu - \lambda^-)\Pi(\nu - \mu^+)}.$$

On the other hand, the fact that the function f is majorized by the pair $P_\pm \in \Pi$ makes it possible to derive the following properties of the function g: a) if $x_i \in S$ $(i = 1, 2)$ and $x_1 \gg x_2$, then

$$g(x_1, \dot{x}_1, \dots, x_1^{(n-1)}) \le g(x_2, \dot{x}_2, \dots, x_2^{(n-1)});$$

b) if $x \in S$, then

$$q_1 a_- = q_2 a_+ \le g(x, \dot{x}, \dots, x^{(n-1)}) \le 0.$$

It is now easy to construct two iterative sequences (one sequence with increasing steepness, the other with decreasing steepness), each of which converges to a transitional solution of equation (1). In this way theorems 1-3 are obtained naturally.

5. Applying theorem 1 to the second-order equation

$$\ddot{x} + f(x, \dot{x}) = 0, \tag{9}$$

we obtain the following assertion:

If the function $f(x, 0)$ has two adjacent zeros $q_1 < 0$ and $q_2 > 0$ and is positive inside the interval (q_1, q_2), while the function $f(x, \dot{x})$ is continuous and differentiable in the region $q_1 \le x \le 0$, $0 \le x \le q_2$, $-\infty < \dot{x} < \infty$, then, for the existence of a transitional solution of equation (9), it is sufficient that the system of inequalities

$$\max_{q_1 \leq x,\, y \leq 0} \{\mu^2 + \mu f'_{\dot{x}}(x, \mu(x-y)) + f'_x(x, \mu(x-y))\} \leq \frac{q_2 - q_1}{q_1}\lambda\mu,$$

$$\min_{0 \leq x,\, z \leq q_2} \{\lambda^2 + \lambda f'_{\dot{x}}(x, \lambda(x-z)) + f'_x(x, \lambda(x-z))\} \geq 0, \qquad \lambda < 0,$$

$$\min_{q_1 \leq x,\, y \leq 0} \{f'_{\dot{x}}(x, \mu(x-y))\} + \lambda + \mu \geq 0, \qquad \lambda + \mu > 0,$$

$$\min_{0 \leq x,\, z \leq q_2} \{f'_{\dot{x}}(x, \lambda(x-z))\} + \lambda + \mu \geq 0.$$

be consistent. With the aid of this assertion one can, for example, show that the equation

$$\ddot{x} - 2(1 + x^2)\dot{x} + 1 - x^2 = 0$$

has a transitional solution. When the function $f(x, 0)$ is negative on the interval (q_1, q_2), an analogous criterion for the existence of a transitional solution can be formulated.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.