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1963

SovietRxiv

Abstract

Full Text

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ON THE ZEROS OF ORTHOGONAL POLYNOMIALS

(Presented by Academician S. N. Bernstein, 4 V 1963)

1. Let $\alpha(x)$ be a given nondecreasing function with an infinite number of points of increase on the interval $[a, b]$, and let the algebraic polynomials $\{p_n(x)\}$ form an orthogonal system on $[a, b]$ corresponding to the distribution $d\alpha(x)$. It is known that all zeros of the polynomial $p_n(x)$ are real and simple and lie inside the interval (a, b); moreover, the zeros of the polynomial $p_n(x)$ interlace with the zeros of the polynomial $p_{n-1}(x)$. We shall generalize this theorem to the case where a sequence of continuously differentiable orthogonal functions $\{\varphi_j(x)\}_{j=0}^{\infty}$ is such that $\{\varphi_j(x)\}_{j=0}^{n}$ forms a Chebyshev system for every natural n. Recall that, by definition (^{1}), a sequence of functions $\{\psi_j(x)\}_{j=0}^{n}$ continuous on $[a, b]$ forms a Chebyshev system of order n (a T_n-system) on $[a, b]$ if every nonidentically zero polynomial

$$P_n(x) = \sum_{j=0}^{n} a_j \psi_j(x) \tag{1}$$

can have on $[a, b]$ no more than n zeros.

We shall adopt the following convention for counting the multiplicity of zeros of functions (cf. (^{1}), p. 8). If a continuously differentiable function $f(x)$ is such that it has on the interval $[a, b]$ under consideration a finite number of zeros, then a point x_0 is regarded as a **simple** zero of the function $f(x)$ if $f(x_0) = 0$, $f'(x_0) \neq 0$; but if $f(x_0) = f'(x_0) = 0$, then the point x_0 is regarded as a **double** zero if, in passing through x_0, the function $f(x)$ does not change sign, and if the function $f(x)$ changes sign in passing through x_0, then the point x_0 is regarded as a **triple** zero. We note, however, that all the arguments and conclusions given below remain valid also in the case where the functions considered are differentiable a sufficient number of times and the multiplicity of zeros is counted in the usual way.

Theorem 1. *Let a sequence of functions $\{\varphi_j(x)\}_{j=0}^{\infty}$, continuously differentiable on the interval $[a, b]$, form an orthogonal system corresponding to the given distribution $d\alpha(x)$, and let, moreover, $\{\varphi_j(x)\}_{j=0}^{n}$ form a T_n-system for every natural n. Then, whatever real number λ may be, the function*

$$\Phi(x; \lambda) = \varphi_n(x) + \lambda \varphi_{n-1}(x) \tag{2}$$

has on $[a, b]$ either n or $n-1$ zeros, and all of them are simple. The function $\varphi_n(x)$ has inside the interval (a, b) n simple zeros, which interlace with the zeros of $\varphi_{n-1}(x)$.

An analogous theorem on the interlacing of the zeros of two consecutive polynomials of least deviation from zero in the metric of the space C was proved in my note (²). In the case of algebraic polynomials, the proof of Theorem 1 relies essentially on the possibility of factoring these polynomials into factors (see (³,⁴)). In the general case, the following theorem will serve as our tool.

Theorem 2. Let $U(x)$ and $V(x)$ be continuously differentiable functions on the interval $[a, b]$ such that the function $U(x)$ has n simple zeros on $[a, b]$, while the function $V(x)$ has either n or $n-1$ simple zeros on $[a, b]$, and, for every real λ, the function

$$F(x; \lambda) = U(x) + \lambda V(x)$$

has $\leqslant n$ zeros on $[a, b]$. In order that the zeros of the functions $U(x)$ and $V(x)$ alternate, it is necessary and sufficient that, for every real λ, the function $F(x; \lambda)$ have only simple zeros.

Denote by $x_1 < x_2 < \cdots < x_n$ the zeros of $U(x)$, and by $\xi_1 < \xi_2 < \cdots < \xi_n$ the zeros of $V(x)$. Suppose that the zeros of $U(x)$ and $V(x)$ alternate, i.e.

$$a \leqslant x_1 < \xi_1 < x_2 < \xi_2 < \cdots < x_{n-1} < \xi_{n-1} < x_n < \xi_n \leqslant b. \tag{3}$$

By virtue of the inequalities (3), the function $V(x)$ changes sign in the interval (x_k, x_{k+1}), so that for every $\lambda \neq 0$ we have

$$F(x_k; \lambda) F(x_{k+1}; \lambda) = \lambda^2 V(x_k) V(x_{k+1}) < 0.$$

We see that the function $F(x; \lambda)$ changes sign in (x_k, x_{k+1}); consequently, the function $F(x; \lambda)$ has an odd number of zeros in (x_k, x_{k+1}). Since the number of such intervals is $n-1$, and $F(x; \lambda)$ can have $\leqslant n$ zeros on $[a, b]$, in each interval (x_k, x_{k+1}) there lies one and only one simple zero of the function $F(x; \lambda)$. The function $F(x; \lambda)$ can have one more simple zero in $[a, x_1)$ or in $(x_n, b]$.

Suppose now that, for every λ, $F(x; \lambda)$ has on $[a, b]$ only simple zeros. This means that for no value of x, $a \leqslant x \leqslant b$, is the system of homogeneous equations

$$F(x; \lambda) = U(x) + \lambda V(x) = 0,$$
$$F'(x; \lambda) = U'(x) + \lambda V'(x) = 0,$$

consistent; consequently, the determinant of this system does not vanish,

$$\Delta(x) = U(x)V'(x) - U'(x)V(x), \qquad a \leqslant x \leqslant b.$$

For definiteness, suppose that $\Delta(x) < 0$. We have $U(x_k) = U(x_{k+1}) = 0$, and, since these zeros are simple, it follows that

$$U'(x_k)U'(x_{k+1}) < 0. \tag{4}$$

On the other hand, since $\Delta(x) > 0$, we have

$$\Delta(x_k) = -U'(x_k)V(x_k) > 0, \qquad \Delta(x_{k+1}) = -U'(x_{k+1})V(x)_{k+1} > 0. \tag{5}$$

From (4) and (5) it follows that

$$V(x_k)V(x_{k+1}) < 0. \tag{6}$$

Consequently, the function $V(x)$ has an odd number of zeros in (x_k, x_{k+1}), and since the number of these intervals is $n-1$, each of them contains exactly one zero of the function $V(x)$, which means that the zeros of the functions $U(x)$ and $V(x)$ alternate.

To prove Theorem 1, let us first establish that the function $\varphi_n(x)$ $(n = 1, 2, \ldots)$ has n simple zeros on $[a, b]$. It is clear that the function $\varphi_n(x)$ must change sign at least once inside (a, b), since

$$\int_a^b \varphi_n(x)\varphi_0(x)\, d\alpha(x) = 0,$$

and the function $\varphi_0(x)$ does not vanish on $[a, b]$, since it constitutes a T_0-system. Suppose that the function $\varphi_n(x)$ changes sign on $[a, b]$ k times, $k < n$. Denote the corresponding zeros of the function $\varphi_n(x)$ by $x_1, x_2, \ldots, x_k$. Construct the polynomial

$$P_k(x) = \begin{vmatrix} \varphi_0(x_1) & \varphi_1(x_1) & \ldots & \varphi_k(x_1) \\ \varphi_0(x_2) & \varphi_1(x_2) & \ldots & \varphi_k(x_2) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_0(x_k) & \varphi_1(x_k) & \ldots & \varphi_k(x_k) \\ \varphi_0(x) & \varphi_1(x) & \ldots & \varphi_k(x) \end{vmatrix}, \tag{7}$$

which has k simple zeros at the points $x_1, x_2, \ldots, x_k$. From the orthogonality conditions it follows that

$$\int_a^b \varphi_n(x) P_k(x)\, d\alpha(x) = 0,$$

which is impossible, since the product $\varphi_n(x)P_k(x)$ does not change sign on $[a,b]$. Thus, the function $\varphi_n(x)$ has n simple zeros on $[a,b]$.

It is proved similarly that for any λ the function $\Phi(x;\lambda)$ has only simple zeros. Indeed, if $\Phi(x;\lambda)$ had at least one double zero, then the number of zeros of the function $\Phi(x;\lambda)$ at which it changes sign would be $k \le n-2$, since $\Phi(x;\lambda)$ is a polynomial in the T_n-system and, consequently, the total number of its zeros is $\le n$. On the other hand, the function $\Phi(x;\lambda)$ is orthogonal to any polynomial of order $k \le n-2$, so that, in particular,

$$\int_a^b \Phi(x;\lambda) P_k(x)\, d\alpha(x) = 0, \tag{8}$$

where $P_k(x)$ is the polynomial defined by (7), and $x_1, x_2, \dots, x_k$ are those zeros of $\Phi(x;\lambda)$ at which this function changes sign. But for the polynomial $P_k(x)$ equality (8) is impossible, since the product $\Phi(x;\lambda)P_k(x)$ does not change sign on $[a,b]$. Consequently, the function $\Phi(x;\lambda)$ has $\ge n-1$ simple zeros on $[a,b]$.

On the basis of Theorem 2 we conclude from this that the zeros of the functions $\varphi_n(x)$ and $\varphi_{n-1}(x)$ mutually interlace.

2. We indicate another application of Theorem 2, namely we establish the following theorem.

Theorem 3. Let

$$P(x) = \prod_{\nu=1}^{m} (x-\alpha_\nu) = \sum_{k=0}^{m} a^k_{kx}, \qquad Q(x) = \prod_{\nu=1}^{m} (x-\beta_\nu) = \sum_{k=0}^{m} b^k_{kx} \tag{9}$$

be algebraic polynomials all of whose zeros are real and mutually interlace; let $u(x)$ and $v(x)$ be continuously differentiable functions on the interval $[a,b]$, having n simple zeros on $[a,b]$ which mutually interlace, and, moreover, let the function $f(x;\lambda) = u(x) + \lambda v(x)$ have $\le n$ zeros on $[a,b]$ for every real λ. Then all zeros of the functions

$$p(x) = \sum_{k=0}^{m} a^k_{ku}(x) v^{m-k}(x), \qquad q(x) = \sum_{k=0}^{m} b^k_{ku}(x) v^{m-k}(x), \tag{10}$$

lying on $[a,b]$, are simple and mutually interlace.

This result generalizes a theorem of P. Montel (^{5}), in which algebraic polynomials of degree n are considered instead of the functions $u(x)$ and $v(x)$.

For the proof, consider the algebraic polynomial $R(x;\lambda) = P(x) + \lambda Q(x)$. By Theorem 2 it has either m or $m-1$ simple real ···

real zeros, so that

$$R(x;\lambda) = \prod_{\nu=1}^{t} (x - \gamma_\nu), \tag{11}$$

where all γ_ν are real and distinct, and $t = m$ or $m-1$. Taking into account (9), (10), and (11), we can write

$$p(x) = \prod_{\nu=1}^{m} [u(x) - \alpha_\nu v(x)], \qquad q(x) = \prod_{\nu=1}^{m} [u(x) - \beta_\nu v(x)];$$

$$r(x;\lambda) = p(x) + \lambda q(x) = \prod_{\nu=1}^{t} [u(x) - \gamma_\nu v(x)]. \tag{12}$$

Since each of the factors occurring on the right-hand sides of (12) is a function of the form $f(x;\mu)$, it has only simple zeros on $[a,b]$, their number being either n or $n-1$. Moreover, no two factors have common zeros. Consequently, each of the functions $p(x)$, $q(x)$, $r(x;\lambda)$ has $\leq mn$ zeros on $[a,b]$, all of them simple. By Theorem 2 it follows from this that the zeros of $p(x)$ and $q(x)$ interlace on $[a,b]$.

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Received
17 IV 1963

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Note: Figure translations are in progress. See original paper for figures.

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