



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

A. M. Bukhvalov

1963

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Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1963. Volume 148, No. 5

MATHEMATICS

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NOMOGRAPHY OF EQUATIONS OF THE FIRST GENUS WITH THREE VARIABLES

(Presented by Academician A. A. Dorodnitsyn, 20 IV 1962)

An equation of the 1st genus is an equality

$$F(u_1, u_2, u_3) = 0, \quad (1)$$

which can be represented by a nomogram of aligned points with one curvilinear and two rectilinear scales. Questions connected with the nomography of equations of the 1st genus have been considered in works ⁽¹⁻²⁾ and others. In the present note a new method is set forth for the nomography of these equations.

It is assumed that in the parallelepiped $U\{\bar{u}_1 \leq u_1 \leq \bar{u}_1, \bar{u}_2 \leq u_2 \leq \bar{u}_2, \bar{u}_3 \leq u_3 \leq \bar{u}_3\}$ equation (1) defines the functions

$$u_\beta = \theta_\beta(u_\alpha, n_\gamma), \quad (2a)$$

$$u_\gamma = \theta_\gamma(u_\alpha, u_\beta). \quad (2b)$$

Here and in what follows the indices α, β, γ form a permutation of the numbers 1, 2, 3.

We consider the differential equations

$$\frac{du_\beta}{du_\alpha} = g_\beta(u_\alpha, u_\beta), \quad (3a)$$

$$\frac{du_\gamma}{du_\alpha} = g_\gamma(u_\alpha, u_\gamma), \quad (3b)$$

which describe the families of level lines of the functions (2b) and (2a), respectively, and are obtained by eliminating u_γ and u_β from equation (1). The functions $g_\beta(u_\alpha, u_\beta)$, $g_\gamma(u_\alpha, u_\gamma)$ are assumed to be sufficiently smooth.

The dependence between the variables u_1, u_2, u_3 , represented by an arbitrary nomogram with a curvilinear scale (u_α) and rectilinear scales (u_β) and (u_γ), after a suitable projective transformation of the plane of the nomogram can always be expressed by Massau's equation of the form

$$\begin{vmatrix} f_\alpha & \varphi_\alpha & 1 \\ f_\beta & 0 & 1 \\ f_\gamma & 1 & 1 \end{vmatrix} = 0, \quad (4)$$

where $f_\alpha = f_\alpha(u_\alpha)$, $\varphi_\alpha = \varphi_\alpha(u_\alpha)$, $f_\beta = f_\beta(u_\beta)$, $f_\gamma = f_\gamma(u_\gamma)$. Thus the problem of nomographing equation (1) is reduced to the problem of bringing this equation to the form (4).

In what follows it is assumed that equations (1) and (4) are equivalent, i.e., upon substitution of $u_\beta = \theta_\beta(u_\alpha, u_\beta)$ (2a) or $u_\gamma = \theta_\gamma(u_\alpha, u_\gamma)$ (2b), they simultaneously become identities. Consequently, the right-hand side of each of the differential equations

$$\frac{du_\beta}{du_\alpha} = \frac{\varphi'_\alpha f_\beta + f'_\alpha \varphi_\alpha - f_\alpha \varphi'_\alpha}{\varphi_\alpha(1 - \varphi_\alpha) f'_\beta}, \quad (5)$$

$$\frac{du_\gamma}{du_\alpha} = -\frac{\varphi'_\alpha f_\gamma + f'_\alpha \varphi_\alpha - f_\alpha \varphi'_\alpha - f'_\alpha}{\varphi_\alpha(\varphi_\alpha - 1) f'_\gamma}, \quad (6)$$

obtained from equation (4), must be identically equal to the corresponding one of the functions $g_\beta(u_\alpha, u_\beta)$ or $g_\gamma(u_\alpha, u_\gamma)$.

If in these equations the dependent variables are taken to be

$$f_\beta = f_\beta(u_\beta), \quad (7a)$$

$$f_\gamma = f_\gamma(u_\gamma), \quad (7b)$$

then they are reduced to linear equations.

It can be proved that the equation

$$\frac{du_\mu}{du_\alpha} = g_\mu(u_\alpha, u_\mu), \quad (8)$$

provided the conditions

$$\begin{vmatrix} g_\mu & g'_\mu & g''_\mu \\ g_\mu^* & g_{\mu'}^* & g_{\mu''}^* \\ g_\mu^{**} & g_{\mu'}^{**} & g_{\mu''}^{**} \end{vmatrix} \equiv 0, \quad * = \frac{\partial}{\partial u_\alpha}, \quad ' = \frac{\partial}{\partial u_\mu}; \quad (9)$$

$$\left(\frac{\delta_{\mu 32}}{\delta_{\mu 33}} \right)' \equiv -\frac{\delta_{\mu 31}}{\delta_{\mu 33}}; \quad (10a)$$

$$\delta_{\mu 33} \neq 0, \quad (10b)$$

are satisfied, where $g_\mu = g_\mu(u_\alpha, u_\mu)$ and $\delta_{\mu ij} = \delta_{\mu ij}(u_\alpha, u_\mu)$ is the minor located in the i -th row and j -th column of the determinant (9), after introduction of the new dependent variable

$$f_\mu = C_\mu I_\mu(u_\mu) = C_\mu \int_{\bar{u}_\mu}^{u_\mu} \frac{du_\mu}{\delta_{\mu 33}(u_\alpha, u_\mu)} \quad (11)$$

(C_μ is an arbitrary nonzero constant) is reduced to a first-order linear differential equation. It can further be proved that, in order for equations (3a) and (3b) to satisfy conditions (9), (10), it is sufficient that one of them satisfy these conditions.

In view of the above, equality (11), for $\mu = \beta, \gamma$, will determine the functions f_β and f_γ entering equation (4).

After determination of the functions f_β and f_γ , the functions f_α and φ_α can be found by various methods and, in particular, in the following way.

In the identities that are obtained after substituting functions (2) into equation (4), one should set $u_\beta = \bar{u}_\beta$, $u_\gamma = \bar{u}_\gamma$ and take into account that, by definition (11), $f_\beta(\bar{u}_\beta) = 0$ and $f_\gamma(\bar{u}_\gamma) = 0$. In this way two equalities are obtained, from which it follows that

$$f_\alpha(u_\alpha) = \frac{C_\beta C_\gamma I_\beta(\theta_\beta(u_\alpha, \bar{u}_\gamma)) I_\gamma(\theta_\gamma(u_\alpha, \bar{u}_\beta))}{C_\beta I_\beta(\theta_\beta(u_\alpha, \bar{u}_\gamma)) + C_\gamma I_\gamma(\theta_\gamma(u_\alpha, \bar{u}_\beta))}, \quad (12)$$

$$\varphi_\alpha(u_\alpha) = \frac{C_\beta I_\beta(\theta_\beta(u_\alpha, \bar{u}_\gamma))}{C_\beta I_\beta(\theta_\beta(u_\alpha, \bar{u}_\gamma)) + C_\gamma I_\gamma(\theta_\gamma(u_\alpha, \bar{u}_\beta))}. \quad (13)$$

Equations (11)–(13) completely determine the elements of equation (4) and exhaust the solution of the problem.

The identities (9), (10a) and inequality (10b) constitute sufficient conditions for representing equation (1) by a nomogram with a curvilinear scale (u_α) and rectilinear scales (u_β), (u_γ). In concrete cases the possibility of such nomographing

of equation (1) is established more simply. Having found an admissible transformation of the dependent variable in one of equations (3), it is necessary to check whether this equation is indeed reduced in this way to a linear one.

Example. The well-known Cauchy equation

$$K(u_\alpha, u_\beta, u_\gamma) \equiv r_\alpha(u_\alpha)r_\beta(u_\beta) + s_\alpha(u_\alpha)r_\gamma(u_\gamma) + t_\alpha(u_\alpha) = 0 \quad (14)$$

(in what follows, where admissible, the arguments will be omitted) determines in some domain U the functions

$$u_\beta = \tilde{r}_\beta \left(-\frac{s_\alpha r_\gamma + t_\alpha}{r_\alpha} \right), \quad (15a)$$

$$u_\gamma = \tilde{r}_\gamma \left(-\frac{r_\alpha r_\beta + t_\alpha}{s_\alpha} \right) \quad (15b)$$

($\tilde{r}_\beta, \tilde{r}_\gamma$ are functions inverse to r_β and r_γ).

The differential equations of the families of level lines of the functions (15) have the following form:

$$\frac{du_\gamma}{du_\alpha} = \frac{(r'_\alpha s_\alpha - r_\alpha s'_\alpha)r_\gamma + r'_\alpha t_\alpha - r_\alpha t'_\alpha}{r_\alpha s_\alpha r'_\gamma}, \quad (16a)$$

$$\frac{du_\beta}{du_\alpha} = \frac{(r'_\alpha s'_\alpha - r'_\alpha s_\alpha)r_\beta + s'_\alpha t_\alpha - s_\alpha t'_\alpha}{r_\alpha s_\alpha r'_\beta}. \quad (16b)$$

If, as the new dependent variables, one takes

$$f_\gamma = C_\gamma(r_\gamma(u_\gamma) - r_\gamma(\bar{u}_\gamma)), \quad (17a)$$

$$f_\beta = C_\beta(r_\beta(u_\beta) - r_\beta(\bar{u}_\beta)), \quad (17b)$$

then equations (16) are reduced to linear ones and, consequently, the functions (17) are the corresponding elements in Massau's equation (4). The functions f_α and φ_α are found by means of formulas (12), (13).

The Massau equation, equivalent to equation (14), has the form

$$\begin{vmatrix} -\frac{C_\beta C_\gamma K(u_\alpha, \bar{u}_\beta, \bar{u}_\gamma)}{C_\beta s_\alpha(u_\alpha) + C_\gamma r_\alpha(u_\alpha)} & \frac{C_\beta s_\alpha(u_\alpha)}{C_\beta s_\alpha(u_\alpha) + C_\gamma r_\alpha(u_\alpha)} & 1 \\ C_\beta(r_\beta(u_\beta) - r_\beta(\bar{u}_\beta)) & 0 & 1 \\ C_\gamma(r_\gamma(u_\gamma) - r_\gamma(\bar{u}_\gamma)) & 1 & 1 \end{vmatrix} = 0. \quad (18)$$

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Submitted
26 XII 1961

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