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Abstract

Full Text

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ON SIMPLE SOLUTIONS OF OPERATOR EQUATIONS IN THE CASE OF BRANCHING

(Presented by Academician V. I. Smirnov on 13 V 1963)

The paper studies the branching problem under very general assumptions. The concept of a simple (r -simple) solution of the operator equation under consideration is introduced; necessary and sufficient conditions for its existence are found; it is shown that the number of simple solutions is determined by a finite number of operations. For all simple solutions of the given equation, majorant series are constructed, which makes it possible to solve certain quantitative problems (to estimate the radii of convergence of the solution series and their remainder terms); in the literature known to us ^(1,2), problems of this kind were solved only for 0-simple solutions.

The question of nonsimple solutions* is not considered in the paper. We note only that for applied problems in which the solution must change little under small, in some sense, changes of the equation itself, nonsimple solutions have no physical meaning.

Let us introduce notation. X, Y, Y' are complex B -spaces; X is finite-dimensional; λ is a complex number; $X(\lambda), Y(\lambda), Y'(\lambda)$ are spaces of series, convergent in some neighborhood of the point $\lambda = 0$, of the form

$$x(\lambda) = \sum_{k=0}^{\infty} \lambda^k x_k, \quad x_k \in X;$$

$$y(\lambda) = \sum_{k=0}^{\infty} \lambda^k y_k, \quad y_k \in Y;$$

$$y'(\lambda) = \sum_{k=0}^{\infty} \lambda^k y'_k, \quad y'_k \in Y'.$$

$Z = X + Y, \quad Z' = X + Y'$, respectively $Z(\lambda) = X(\lambda) + Y(\lambda), \quad Z'(\lambda) = X(\lambda) + Y'(\lambda)$.

$C(\lambda) = \sum_{k=0}^{\infty} \lambda^k C_k$, where $\{C_k\}$ is a set of linear operators defined on Z with range in Z' and such that the series $C(\lambda)$ converges for sufficiently small λ . Obviously, $C(\lambda)[Z(\lambda)] \supset Z'(\lambda)$.

A linear operator A is defined on Z and has the following properties: $A(Z) = Y'$; $A(X) = 0$; there exists a bounded inverse A^{-1} from Y' to Y . $A^{-1}(Y') = Y$.

The operators α and β project Z' onto X and Y' , respectively; α_1 projects Z onto X .

$C_N(x, z, \lambda) = C(\lambda)z_N$, where $\{z_k\}$ is a recursively defined sequence:

$$z_0 = z, \quad z_{k+1} = x + \lambda^h A^{-1} \beta C(\lambda) z_k.$$

$C_\infty(x, \lambda) = \lim_{k \rightarrow \infty} C_k(x, z, \lambda)$. The fact of existence of the linear operator $C_\infty(x, \lambda)$ and its independence of z for sufficiently small λ follows immediately from the Banach principle.

* The difficulty in this question is presented only by the case when the nonsimple solution is isolated. At present not a single equation having such a solution is known.

$B(z, \lambda)$ is an operator analytic in both variables in the domain $Z \times D$, where D is some neighborhood of the point $\lambda = 0$ in the complex plane, and with range in $Z'(\lambda)$.

$B_N(x, z, \lambda) = B(z_N, \lambda)$, where $\{z_k\}$ is a recursively defined sequence:

$$z_0 = z, \quad z_{k+1} = x + \lambda^n A^{-1} \beta B(z_k, \lambda).$$

$$B_\infty(x, \lambda) = \lim_{k \rightarrow \infty} B_k(x, z, \lambda).$$

The operator $B_\infty(x, \lambda)$ exists on the basis of the considerations given concerning the existence of $C_\infty(x, \lambda)$.

Definition 1. Let r be a nonnegative integer. We shall call an r -**solution** a solution of the operator equation which is a function of the parameter λ , representable in a neighborhood of the point $\lambda = 0$ in the form of a Laurent series and having there a pole of order r . In keeping with this terminology, we shall henceforth call a **0-solution** (zero-solution) a solution of the operator equation analytic with respect to λ in a neighborhood of the point $\lambda = 0$.

Theorem 1. In order that the equation

$$A(z) = \lambda^n [z'(\lambda) + C(\lambda)z] \tag{1}$$

have a unique r -solution for every $z'(\lambda) \in Z'(\lambda)$, it is necessary and sufficient that the equation

$$\alpha C_{\infty}(x, \lambda) = x'(\lambda) \quad (2)$$

have a unique r -solution $x(\lambda)$ for every $x'(\lambda) \in X(\lambda)$.

Theorem 2. In order that equation (1) have a unique r -solution for every $z'(\lambda) \in Z'(\lambda)$, it is necessary and sufficient that the equation

$$\alpha C_N(x, x, \lambda) = x'(\lambda) \quad (3)$$

with $N = [r/n]$ have a unique r -solution for every $x'(\lambda) \in X(\lambda)$.

Remark. The number $[r/n]$ in the formulation of Theorem 2 may be replaced by any larger number.

Definition 2. We shall call a 0-solution $z_0(\lambda)$ of the equation*

$$A(z) = \lambda^n B(z, \lambda) \quad (4)$$

r -simple if there exist nonnegative integers r and R_0 such that, for $R \geq R_0$ and for all $z'(\lambda) \in Z'(\lambda)$, the equation

$$A(z) = \lambda^n [z'(\lambda)\lambda^R + B(z, \lambda)]$$

has a unique solution of the form $z_0(\lambda) + \lambda^{R-r} z_1(\lambda)$, where $z_1(\lambda) \in Z(\lambda)$.

Theorem 3. In order that the 0-solution $z_0(\lambda)$ of equation (4) be r -simple, it is necessary and sufficient that the linear equation

$$A(z) = \lambda^n \{z'(\lambda) + B'_z[z_0(\lambda), \lambda]z\}$$

have a unique r -solution for every $z'(\lambda) \in Z'(\lambda)$.

Theorem 4. In order that the 0-solution $z_0(\lambda)$ of equation (4) be r -simple, it is necessary and sufficient that equation (4') with $R = 2r + 1$ and for all $z'(\lambda) \in Z'(\lambda)$ have a unique solution of the form $z_0(\lambda) + \lambda^{r+1} z_1(\lambda)$, where $z_1(\lambda) \in Z(\lambda)$.

Theorem 5. In order that the 0-solution $z_0(\lambda)$ of equation (4) be r -simple, it is necessary and sufficient that: a) the function $\alpha_1 z_0(\lambda) = x_0(\lambda)$ be a 0-solution of the equation $\alpha B_{\infty}(x, \lambda) = 0^{**}$; b) the equation $\alpha B'_{\infty}[x_0(\lambda), \lambda]x = x'(\lambda)$ have a unique r -solution for every $x'(\lambda) \in X(\lambda)$.

Remark. It follows from Theorem 5 that $z_0(\lambda)$ is not a simple solution of equation (4) if and only if the determinant***

* It is easy to see that every equation in the case of branching can be reduced to such a form by a suitable change of variables.

** This equation is, of course, a branching equation.

*** This determinant is the Jacobian of the system of branching equations.

of the matrix $aB'_\infty[x_0(\lambda), \lambda]$ is identically zero. The simplest example of such a case is supplied by any real periodic solution of the differential equation

$$\dot{y} = \lambda \sin t (1 + y^2).$$

In this case, as is not hard to see, $aB_\infty(x, \lambda) \equiv 0$, and the equation has an infinite set of 2π -periodic 0-solutions: $y = \operatorname{tg}(c - \lambda \cos t)$, where c is an arbitrary constant. As mentioned above, we do not have an example of an isolated nonsimple solution.

Theorem 6. Let $N = \left\lfloor \frac{2r+1}{n} \right\rfloor$. To each r -simple 0-solution $z_0(\lambda)$ of equation (4) there corresponds a unique 0-solution $x_0(\lambda)$ of the equation

$$\alpha B_N(x, x, \lambda) = 0,$$

satisfying the conditions: a) the equation $\alpha B'_N(x_0, x_0, \lambda)x = x'(\lambda)$ has a unique r -solution for every $x'(\lambda) \in X(\lambda)$; b) $\alpha z_0(\lambda) - x_0(\lambda) = \lambda^{r+1}x_1(\lambda)$, where $x_1(\lambda) \in X(\lambda)$. This correspondence is one-to-one.

Lemma. The sets of 0-solutions of equation (4) and of the equation

$$A(z) = \lambda^n B_m(\alpha_1 z, z, \lambda),$$

where m is an arbitrary nonnegative number, coincide.

Theorem 7*. Let $x_0^{(r)}(\lambda)$ be the partial sum of the series $x_0(\lambda)$ (of degree λ^r). Represent $B_N[x_0^{(r)}(\lambda) + x', x_0^{(r)}(\lambda) + x' + y, \lambda]$ in the form of a double Γ -series in x' and y :

$$\begin{aligned} & B_N[x_0^{(r)}(\lambda) + x', x_0^{(r)}(\lambda) + x' + y, \lambda] = \\ & = P_{00}(\lambda) + P_{10}(\lambda)x' + \sum_{\substack{i \geq 2 \\ j \geq 1}} P_{ij}(\lambda)(x', y). \end{aligned}$$

Let $\varphi_1(\lambda)$, $\varphi_2(\lambda)$, $f(\bar{x}', \bar{y}, \lambda)$ be power series with nonnegative coefficients such that

$$[\alpha P_{10}(\lambda)]^{-1} \preccurlyeq \lambda^{-r} \varphi_1(\lambda), \quad [\alpha P_{10}(\lambda)]^{-1} [\alpha P_{00}(\lambda)] \preccurlyeq \varphi_2(\lambda),$$

$$\sum_{\substack{i \geq 2 \\ j \geq 1}}^{\infty} P_{ij}(\lambda)(x', y) \approx f(\bar{x}', y, \lambda).$$

Then the formula

$$z_0(\lambda) - x_0^{(r)}(\lambda) \approx \bar{x}'(\lambda) + \bar{y}(\lambda),$$

holds, where $[\bar{x}'(\lambda), \bar{y}(\lambda)]$ is the unique 0-solution of the system of equations

$$\bar{x}' = \varphi_2(\lambda) + \lambda^{-r} \varphi_1(\lambda) f(\bar{x}', y, \lambda),$$

$$\bar{y} = \lambda^n \|A^{-1}\| [\bar{P}_{00}(\lambda) + \bar{P}_{00}(\lambda) \bar{x}' + f(\bar{x}', y, \bar{y})].$$

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- ¹ M. M. Vainberg, V. A. Trenogin, *UMN*, **17**, issue 2 (104) (1962).
² A. E. Gelman, *DAN*, **144**, No. 1 (1962).

* The notation of Theorem 6 is retained.

Note: Figure translations are in progress. See original paper for figures.

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