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Abstract

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MATHEMATICS

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GENERAL BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

(Presented by Academician S. L. Sobolev on 20 VII 1962)

1°. Recently, in a number of works $^{(1-9)}$, boundary-value problems for elliptic equations of the 2nd order with discontinuous coefficients have been studied by various methods.

In $^{(10)}$ the authors considered boundary-value problems and eigenvalue problems for elliptic equations of the 2nd order with discontinuous coefficients by functional methods connected with the use of inequalities of Gårding type.

In the present note the solvability, in the generalized and ordinary sense, of general boundary-value problems for elliptic equations of arbitrary order with discontinuous coefficients is proved; the boundary conditions and conjugation conditions on the surfaces of discontinuity are given by general differential operators*.

In works $^{(11-16)}$ boundary-value problems for equations with continuous coefficients were investigated with the aid of energy inequalities with a boundary norm.

In the present note, problems for equations with discontinuous coefficients are studied with the aid of inequalities of this kind, proved by the authors $^{(17)}$. Here the well-known functional method is used. The notation in this note is the same as in $^{(17)}$.

2°. Let G be a bounded domain of n-dimensional Euclidean space E_n with boundary Γ; G_1 a subdomain of G with boundary γ, having no points in common with Γ; $G_2 = G \setminus \overline{G}_1$**. Introduce the direct sum of Sobolev spaces

$$W_2^l(G_1) \dot{+} W_2^l(G_2) = W_2^l(G) = W_2^l \quad (l \geq 0 \text{ integer});$$

every function $u \in W_2^l$ can be represented in the form $u(x) = u_1(x) + u_2(x)$, where $u_i(x) = u(x)$, $x \in G_i$; $u_i(x) = 0$, $x \in G \setminus \overline{G}_i$ $(i = 1, 2)$; if $l > 0$, $x \in \gamma$, then $u_i(x)$ denotes the limiting value of $u(x)$ from the side of G_i.

Consider the elliptic differential operator A with discontinuous complex coefficients

$$(Au)(x) = \begin{cases} (A^1u)(x), & x \in G_1, \\ (A^2u)(x), & x \in G_2, \end{cases} \tag{1}$$

where

$$A^i = \sum_{|\mu| \le 2m} a^i_\mu(x) D^\mu, \quad x \in G_i \ (i = 1, 2);$$

$$\mu = (\mu_1, \dots, \mu_n); \quad D^\mu = D_1^{\mu_1} \cdots D_n^{\mu_n}; \quad D_k = \frac{1}{i}\frac{\partial}{\partial x}.$$

A^+ is an operator of type (1), formally adjoint to A.

We also introduce **boundary** operators (on Γ) and **conjugation** operators (on γ):

$$B^i_k = \sum_{|\mu| \le m^i_k} b^i_{k\mu}(x) D^\mu \tag{2}$$

$(i = 1, 2, 3; \quad k = 1, \dots, r_i; \quad r_1 = r_2 = 2m, \quad r_3 = m; \quad m^1_k = m^2_k = m_k; \quad m^i_k \le 2m{-}1),$

where the complex functions $b^i_{k\mu}(x)$ $(i = 1, 2)$ are defined on γ, and $b^3_{k\mu}(x)$ on Γ. The system of boundary operators B^3_k is called **normal** (^{14,15}) if all of them are of different orders and Γ is not characteristic for any one of them. The normality of systems of conjugation operators is defined analogously.

Let $W_2^{2m}(\mathrm{gr})$ denote the subspace of functions $u \in W_2^{2m}$ for which

$$[B_{ku}] = B^1_k u_1 - B^2_k u_2 \,\big|_\gamma = 0 \quad (k = 1, \dots, 2m),$$

$$B_{ju} \,\big|_\Gamma = 0 \quad (j = 1, \dots, m). \tag{3}$$

By $W_2^{2m}(\mathrm{gr})^+$ we denote the subspace of all such $v \in W_2^{2m}$ that the equality $(Au, v)_0 = (u, A^+v)_0$ holds for all $u \in W_2^{2m}(\mathrm{gr})$ if and only if $v \in W_2^{2m}(\mathrm{gr})^+$.

Using integration by parts one can show that, if the operator is elliptic, the systems of boundary operators and conjugation operators (2)

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* In the authors' note $^{(10)}$, not only differential operators were allowed as boundary operators.

** All results are also valid for a partition of G into a finite number of domains.

are normal, $a^i_\mu(x) \in C^{|\mu|}(\overline{G_i})$, $b^i_{k\mu}(x) \in C^{\max\{2m-m_k-1,m_k\}}(\gamma)$ $(i = 1,2)$, $b^3_{j\mu}(x) \in C^{\max\{2m-m^3_j-1,m^3_j\}}(\Gamma)$, and Γ and γ are of class C^{2m}, then $W^{2m}_2(\mathrm{gr})^+$ is determined by conditions analogous to (3), and the systems of the corresponding conjugate operators B'^i_k are also normal.

The following lemma plays an essential role below.

Lemma 1. *The normal operators (2) cover* * *the operator* A *if and only if the conjugate operators* B'^i_k $(i = 1,2,3)$ *cover* A^+.

The proof of this lemma is rather cumbersome; for our case it is obtained by developing the algebraic apparatus applied in ([14,18]).

3°. Consider the boundary-value problem

$$Au = f, \qquad u \in W^{2m}_2(\mathrm{gr}) \tag{4}$$

and the conjugate problem

$$A^+v = g, \qquad v \in W^{2m}_2(\mathrm{gr})^+. \tag{5}$$

A function $u \in L_2$ will be called a weak solution of problem (4) if $(u, A^+v)_0 = (f,v)_0$, $v \in W^{2m}_2(\mathrm{gr})^+$. The weak solution of problem (5) is defined analogously.

From the results of ([17]) and Lemma 1 it follows that, if the normal operators (2) cover A, $a^i_\mu(x) \in C^{|\mu|}(\overline{G_i})$, $b^i_{k\mu}(x) \in C^{2m-m^i_k}(\gamma)$ (or Γ) $(i = 1,2,3)$, $b'^i_{k\mu}(x) \in C^{2m-m'^i_k}(\gamma)$ (or Γ), and Γ and γ are of class C^{2m}, then there exist—

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* The definition of covering for the case of discontinuous coefficients is given in ([17]) (Definition 2).

there exists a constant $c > 0$ such that

$$c^{-1}\|u\|^2_{2m} \leqslant B(u,u) + \|u\|^2_0 \leqslant c\|u\|^2_{2m}, \qquad u \in W^{2m}_2; \tag{6}$$

$$c^{-1}\|v\|^2_{2m} \leqslant B^*(v,v) + \|v\|^2_0 \leqslant c\|v\|^2_{2m}, \qquad v \in W^{2m}_2. \tag{7}$$

Here

$$B(u,v) = (Au,Av)_0 + \sum_{k=1}^{2m}\langle [B_k u], [B_k v]\rangle_{2m-m_k-1} + \sum_{j=1}^{m}\langle B^3_j u, B^3_j v\rangle_{2m-m^3_j-1};$$

$B^*(u,v)$ is defined analogously.

Denote by N (N^*) the subspace of solutions in W_2^{2m} of problem (4) with $f = 0$ (respectively, of problem (5) with $g = 0$). From inequalities (6), (7) it follows easily that N and N^* are finite-dimensional.

For the positive space $H_+ = W_2^{2m}$ and the null space $H_0 = L_2$, construct the space with negative norm $H_- = W_2^{-2m}$ (^{10}).

Theorem 1. Suppose the operators A^i, B_k^i are such that inequality (7) holds. If $f \in W_2^{-2m}$, $(f, N^*)_0 = 0$, then there exists a weak solution $u \in L_2$ of problem (4).

Theorem 2. Suppose, in addition to the conditions of Theorem 1, $f \in W_2^s$, γ and Γ are of class C^{4m+s}, $a_\mu^i(x) \in C^{2m+\max\{|\mu|,s\}}(\overline{G}_i)$, $b_{k\mu}^i(x) \in C^{2m+s-1}(\gamma)$ (or Γ) $(i = 1, 2, 3)$. Then the weak solution u found in Theorem 1 belongs to W_2^{2m+s} and, consequently, is a solution in the ordinary sense.

Theorem 1 is proved with the aid of Riesz' s theorem on the representation of a linear functional in a Hilbert space.

Theorem 2 is proved by a method analogous to that used in (^{10,14,19,20}).

From Theorems 1 and 2 one can derive that if the smoothness conditions of Theorem 2 are satisfied, then for $f \in W_2^s$ every weak solution u of problem (4) belongs to W_2^{2m+s}.

4^0. In this subsection the smoothness requirements will be the same as in Theorem 2. Then $N \subset W_2^{2m+s}$. Denote by $M^{2m+s}(\mathrm{gr})$ the set of elements $u \in W_2^{2m}(\mathrm{gr}) \supset W_2^{2m+s}$ for which $(u, N)_0 = 0$. Analogously, denote by M^s the set of those $f \in W_2^s$ for which $(f, N)_0 = 0$. In a similar way one defines $M^{*\,2m+s}(\mathrm{gr})^+$, M^{*s} in $W_2^{2m}(\mathrm{gr})^+$, N^*.

For the positive spaces $W_2'^{s} = M^s$, $W_2'^{\,2m+s} = M^{*\,2m+s}(\mathrm{gr})^+$ and the null space L_2, construct the spaces with negative norm $W_2'^{-s}$, $W_2''^{-2m-s}$.

Theorem 3. Consider the mapping $\Lambda : u \to Au$ $(u \in M^{2m+s}(\mathrm{gr}),\ s \geqslant 0)$ as an operator acting in one of the following pairs of spaces:

$$M^{2m+s}(\mathrm{gr}) \to M^{*s}, \qquad W_2'^{-s} \to W_2''^{-2m-s}.$$

Then for the first pair of spaces Λ, and for the second pair of spaces its closure $\overline{\Lambda}$, is a homeomorphism between the indicated spaces. An analogous assertion holds for the operator A^+.

5^0. Consider the problem with nonhomogeneous boundary conditions and conjugation conditions:

$$Au = f \in W_2^s, \qquad s \geqslant 0;$$

$$[B_k u]_\gamma = \varphi_k \in C^{2m-m_k-s}(\gamma), \qquad k = 1, \dots, 2m; \tag{8}$$

$$B_j^3 u\big|_\Gamma = \psi_j \in C^{2m-m_j{}^3+s}(\Gamma), \qquad j = 1, \dots, m.$$

If the systems of operators (2) are normal, and their coefficients are sufficiently smooth, then one can find a function $u_0 \in W_2^{2m+s}$ such that $B_k u_0\big|_\gamma = \varphi$, $B_j^3 u_0\big|_\Gamma = \psi_j$; then for $w = u - u_0$ we shall have a problem with homogeneous conditions on Γ and γ and with right-hand side $f_1 = f - Au_0$; for this problem everything stated above is valid.

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