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Abstract

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Analytic Continuation of the Scalar Product of Hecke Series of Two Quadratic Fields and Its Application

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1. In the works of Hecke (^{1}) there are introduced the so-called characters of absolute value

$$\lambda(a) = a/|a| = e^{i \arg a},$$

where a is an algebraic number. Hecke poses and solves the problem of the asymptotic distribution of the prime numbers of quadratic fields in homothetically expanding contours. Subsequently these investigations were continued and asymptotic formulas with remainder terms were obtained in the works of H. Rademacher (^{4}) and I. Kubilius (^{2}). To solve these problems one considers the Z-functions introduced by Hecke in (^{1}).

Naturally there arises the question of the distribution of prime numbers of several quadratic fields. Namely: let $K_1, K_2, \dots, K_r$ be r imaginary quadratic fields (assumed, for simplicity, to be one-class fields); to the number $a_l = x_l + iy_l \in K_l$ we shall associate the point of the plane T_l with coordinates (x_l, y_l). It is required to find the number of such systems $(\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r)$ of prime numbers of the fields $K_1, K_2, \dots, K_r$ that

$$N_{K_1/R}\mathfrak{p}_1 = N_{K_2/R}\mathfrak{p}_2 = \cdots = N_{K_r/R}\mathfrak{p}_r$$

and the images $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ lie in homothetically expanding contours $D_1^{(t)}, D_2^{(t)}, \dots, D_r^{(t)}$ (t is the coefficient of similarity). In the case of fields with several classes, instead of prime numbers one must consider ideal prime numbers (see Hecke (^{1})). This problem is connected with the question of the analytic continuation of the scalar product of Hecke series:

$$Z\left(s; \lambda_1^{l_1}, \dots, \lambda_r^{l_r}\right) = \sum_{N_{K_1/R}\alpha_1 = \cdots = N_{K_r/R}\alpha_r} \frac{\lambda_1^{l_1}(\alpha_1) \cdots \lambda_r^{l_r}(\alpha_r)}{\left(N_{K_1/R}\alpha_1\right)^s},$$

where the summation extends over all integers $\alpha_1 \in K_1, \dots, \alpha_r \in K_r$. This formulation of the problem is due to Yu. V. Linnik.

In the present note this question is considered for two quadratic fields $K_1 = R(\sqrt{-1})$ and $K_2 = R(\sqrt{-3})$ (R is the field of rational numbers). The consideration of any other pair of imaginary one-class quadratic fields is analogous. We state the main results.

Theorem 1. The number of pairs of prime numbers $(\mathfrak{p}, \mathfrak{q})$ ($\mathfrak{p} \in R(\sqrt{-1})$; $\mathfrak{q} \in R(\sqrt{-3})$) in the domain

$$\varphi_1 \leq \arg \mathfrak{p} \leq \varphi_2, \qquad \widetilde{\varphi}_1 \leq \arg \mathfrak{q} \leq \widetilde{\varphi}_2, \qquad N_{K_1/R}\mathfrak{p} = N_{K_2/R}\mathfrak{q} \leq X,$$

is equal to

$$\left(\frac{6}{\pi^2}\right) \int_2^X \frac{du}{\log u} (\varphi_2 - \varphi_1)(\widetilde{\varphi}_2 - \widetilde{\varphi}_1) + O\left(X e^{-c_1\sqrt{\log X}}\right).$$

Theorem 2. Let $D_1 : r_1 = f_1(\varphi_1)$ and $D_2 : r_2 = f_2(\varphi_2)$ be two smooth closed contours in the planes $T_1 = \{(x, y)\}$, $T_2 = \{(z, t)\}$ ((r_i, φ_i) are polar coordinates on T_1 and T_2). Let $C_1^{(t)}$ and $C_2^{(t)}$ be contours similar to C_1 and C_2 (the center of similarity is the origin, the coefficient of similarity is t); $Q_1^{(t)}$ and $Q_2^{(t)}$ are the domains bounded by $C_1^{(t)}$ and $C_2^{(t)}$. Then the number of pairs of prime numbers $(\mathfrak{p}, \mathfrak{q})$ ($\mathfrak{p} \in R(\sqrt{-1})$; $\mathfrak{q} \in R(\sqrt{-3})$) such that $\mathfrak{p} \in Q_2^{(t)}$, $\mathfrak{q} \in Q_2^{(t)}$, $N_{K_1/R}\mathfrak{p} = N_{K_2/R}\mathfrak{q}$, is equal to

$$\left(\frac{6}{\pi^2}\right) \left(\int_0^{2\pi} \int_0^{2\pi} \min\{f_1^2(\varphi_1), f_2^2(\varphi_2)\}\, d\varphi_1\, d\varphi_2\right) \frac{t^2}{\log t} + O\left(\frac{t^2}{\log^2 t}\right).$$

The main step in the proof of these theorems is the following lemma, which apparently is also of some independent interest.

Lemma 1. Put $f(x) = x_1^2 + x_2^2 - x_3^2 - 3x_4^2$; the number of integral points on the surface $f(x) = 0$ in the region $x_1, x_2, x_3, x_4 \geqslant 0$; $\operatorname{tg}\varphi \leqslant x_2/x_1 \leqslant \operatorname{tg}(\varphi + \Delta_1)$; $\operatorname{tg}\widetilde{\varphi} \leqslant x_4/x_3 \leqslant \operatorname{tg}(\widetilde{\varphi} + \Delta_2)$; $x_1^2 + x_2^2 \leqslant X$, under the condition $\Delta_1 > X^{-1/250}$, $\Delta_2 > X^{-1/250}$, is equal to

$$N^X_{\varphi_1,\varphi_2}(\Delta_1, \Delta_2) = C X \Delta_1 \Delta_2 \left(\frac{6}{\pi^2}\right) + O\left(X^{1-1/500}\right),$$

where

$$C = \lim_{X \to \infty} \frac{N(X)}{X},$$

and $N(X)$ is the number of points on the surface $f(x)=0$ in the region $x_1^2+x_2^2 \leqslant X,\ x_1,x_2,x_3,x_4 \geqslant 0,\ 0 \leqslant x_4/x_3 \leqslant \sqrt{3}$.

In § 2 we shall show how, from Lemma 1, by means of analytic continuation of the scalar product of Hecke series, the theorems can be proved, and in § 3 we shall outline the plan of the proof of Lemma 1.

2. Let $\alpha \in K_1 = R(\sqrt{-1})$; $\beta \in K_2 = R(\sqrt{-3})$. We call characters the functions

$$\lambda(\alpha) = \frac{\alpha}{|\alpha|} = e^{i \operatorname{arc} \alpha}, \qquad \mu(\beta) = \frac{\beta}{|\beta|} = e^{i \operatorname{arc} \beta},$$

and let

$$Z(s,\lambda^{4n}) = \sum_{\alpha \in K_1} \frac{\lambda^{4n}(\alpha)}{(N_{K_1/R}\alpha)^s} \quad \text{and} \quad Z(s,\mu^{6m}) = \sum_{\beta \in K_2} \frac{\mu^{6m}(\beta)}{(N_{K_2/R}(\beta))^s}$$

(the summation is over the integers $\alpha \in K_1,\ \beta \in K_2$, lying in the regions $0 \leqslant \operatorname{arc}\alpha \leqslant \pi/2,\ 0 \leqslant \operatorname{arc}\beta \leqslant \pi/3$). These functions were first introduced and studied by Hecke in (1).

Consider the scalar product of Hecke series:

$$Z(s,\lambda^{4n},\mu^{6m}) = \sum_{N_{K_1/R}\alpha = N_{K_2/R}\beta} \frac{\lambda^{4n}(\alpha)\mu^{6m}(\beta)}{(N_{K_1/R}\alpha)^{2s}} =$$

$$= \prod_{p \neq 1 \pmod{12}} \left(1-\frac{1}{p^{4s}}\right)^{-1} \cdot \prod_{N_{K_1/R}\mathfrak{p} = N_{K_2/R}\mathfrak{q}} \left(1-\frac{\lambda^{4n}(\mathfrak{p})\mu^{6m}(\mathfrak{q})}{(N_{K_1/R}\mathfrak{q})^{2s}}\right)^{-1}$$

(where the sum extends over all integers, and the product over all prime numbers of the region $0 \leqslant \operatorname{arc}\alpha \leqslant \pi/2,\ 0 \leqslant \operatorname{arc}\beta \leqslant \pi/3$, p is a rational prime number). It follows directly from Lemma 1 that $Z(s,\lambda^{4n},\mu^{6m})$ is a function analytic for $\operatorname{Re} s > 1/2 - 1/3000,\ nm \neq 0$.

Indeed,

$$A_X = \sum_{N_{K_1/R}\alpha = N_{K_2/R}\beta \leqslant X} \lambda^{4n}(\alpha)\mu^{6m}(\beta) =$$

$$= \sum_{k=0}^{[\frac{\pi}{2}X^{1/1500}]} \sum_{l=0}^{[\frac{\pi}{3}X^{1/1500}]} e^{4nikX^{-1/1500}+6milX^{-1/1500}} N_{kX^{-1/1500},\,lX^{-1/1500}}(X^{-1/1500},X^{-1/1500})+$$

$$+O\left(\sum_{k,l} N_{kX^{-1/1500},\,lX^{-1/1500}}(X^{-1/1500}, X^{-1/1500}) \cdot \left|1 - e^{iX^{-1/1500}(4n+6m)}\right|\right) =$$

$$= \int_0^{\pi/2}\int_0^{\pi/3} e^{4ni\varphi+6mi\widetilde{\varphi}}\, d\varphi\, d\widetilde{\varphi} \cdot CX + O\left(X^{1/500}\right) + O\left(X^{1-1/750}X^{1/750-1/1500}\right) = O\left(X^{1-1/1500}\right).$$

$$Z(s, \lambda^{4u}, \mu^{6m}) = \sum_{NK_1/R^\alpha = NK_2/R^\beta} \frac{\lambda^{4n}(\alpha)\mu^{6m}(\beta)}{(NK_1/R^\alpha)^{2s}} =$$

$$= \sum_{X=1}^{\infty} \frac{A_X - A_{X-1}}{X^{2s}} = \sum_{X=1}^{\infty} A_X \left(\frac{1}{X^{2s}} - \frac{1}{(X+1)^{2s}}\right) =$$

$$= O\left(\sum_{X=1}^{\infty} A_X \frac{2s}{X^{2s+1}}\right) = O\left(\sum_{X=1}^{\infty} \frac{1}{X^{2s+1/1500}}\right),$$

where the last series converges absolutely for $\operatorname{Re} s > 1/2 - 1/3000$. Similarly, it is not difficult to prove the following lemma.

Lemma 2. In the domain $1/2 - 1/3000 \leqslant \operatorname{Re} s \leqslant 2$,

$$|Z(s, \lambda^{4n}, \mu^{6m})| < c_2(1 + |n| + |m|)^2(1 + |t|),$$

where $t = \operatorname{Im} s$.

Hence, as usual (see, for example, $(^2)$), Lemma 3 is obtained.

Lemma 3. There exist constants $c_5, c_3, c_4 > 1$ such that $Z(s, \lambda^{4n}, \mu^{6m})$ has no zeros in the domain

$$\operatorname{Re} s \geqslant \begin{cases} 1 - \dfrac{1}{c_3 \log(1+|n|+|m|)\log(1+|t|)}, & |t| \geqslant c_5, \\ 1 - \dfrac{1}{c_4 \log(1+|n|+|m|)(1+c_5)}, & |t| \leqslant c_5. \end{cases}$$

Now Theorem 1 is obtained with the aid of one lemma of I. M. Vinogradov in the same way as Theorem 9 in paper $(^2)$. Theorem 2 is a simple consequence of Theorem 1.

3. To prove Lemma 1 one applies the Hardy–Littlewood method and a generalization of I. M. Vinogradov' s lemma on the expansion into a Fourier series of smoothed characteristic functions of a set (Remark 9, Ch. III, § 4 of monograph $(^3)$).

First one obtains the asymptotic formula

$$\sum_{f(x)=0} e^{-\nu g(x)+2\pi i(\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3+\lambda_4 x_4)} =$$

$$= K\frac{1}{\nu}\left(\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{\exp\left(-\frac{2\pi^2}{\nu}\left(\frac{\lambda_1^2+\lambda_2^2}{4-t}+\frac{\lambda_3^2+\frac{1}{3}\lambda_4^2}{t}\right)\right)}{t(4-t)}\,dt\right)+$$

$$+O\left(\left(\frac{1}{\nu}\right)^{-3/4+11\eta/2+\varepsilon}\right),$$

where

$$\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2}\leqslant \nu^{1/2-\eta}, \qquad \varepsilon>0; \quad \nu\to 0,$$

$$K=\frac{2\pi^2}{\sqrt{3}}\sum_{q=1}^{\infty}\frac{\chi_{12}(q)\varphi(q)}{q^2}, \qquad \chi_{12}(q)=\begin{cases}1, & q\equiv\pm 1 \pmod{12},\\ -1, & q\equiv\pm 5 \pmod{12},\\ 0, & (q,12)\neq 1;\end{cases}$$

$\varphi(q)$ is the Euler function; $f(x)=x_1^2+x_2^2-x_3^2-3x_4^2$, $g(x)=x_1^2+x_2^2+x_3^2+3x_4^2$.

The plan for deriving this formula is the same as the plan for the proof of Theorem 1, § 3, Ch. III of monograph ([3]). Hence Lemma 1 is obtained by a method similar to the proof of Theorem 3, § 4, Ch. III of the monograph mentioned above.

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