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Abstract

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ON THE SOLVABILITY OF LIE ALGEBRAS WITH A REGULAR AUTOMORPHISM OF FINITE PERIOD

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1. An automorphism Φ of a Lie algebra (ring) $\mathfrak{L}$ is called **regular** if $\Phi(x) \neq x$ for all $x \neq 0,\ x \in \mathfrak{L}$. In the present note the following assertion is proved.

Theorem. *A Lie algebra $\mathfrak{L}$ of arbitrary dimension over a field of characteristic $p \geqslant 0$, admitting a regular automorphism of finite period n ($\Phi^n = 1$), is solvable, and the length of its derived series does not exceed 2^{n-1}.*

From this theorem and Theorem B (^{3}) the following result is obtained.

Corollary. *A Lie ring $\mathfrak{L}$ with a regular automorphism of prime period q is nilpotent, and its nilpotency class does not exceed*

$$\frac{(q-1)^{2^{q-1}}-1}{q-2}.$$

The nilpotency of a Lie ring with a regular automorphism of period q was proved earlier by Higman (^{1}). In the same work it was proved that the nilpotency class of such a Lie ring does not exceed some number $k(q)$, depending only on q. From the results of Thompson (^{2}) and Higman (Theorem 3 (^{1})) it follows that the number $k(q)$ is also an upper bound for the nilpotency class of a finite group with a regular automorphism of prime period q. The values of $k(q)$ for $q = 2, 3, 5$ are respectively equal to $1, 2, 6$. For $q \geqslant 7$ it is known only that $k(q) \geqslant (q^2-1)/4$. The corresponding example was given by Higman (^{1}). The corollary stated above gives a certain upper bound for the nilpotency class of a Lie ring (and consequently also of a finite group) with a regular automorphism of period q, although, undoubtedly, this bound is considerably larger than the number $k(q)$. To obtain the exact value of $k(q)$ for all prime q, more refined methods are apparently necessary.

2. The situation changes substantially if the Lie algebra has a regular automorphism of arbitrary period n.

First, such an algebra may be solvable but nonnilpotent. We give an example for any composite n. Let $n = q \cdot m$, where q is some prime divisor of the number n, $m \neq 1$. Consider the Lie algebra $\mathfrak{L}$ generated by the elements $x, y_{1+iq},\ i = 0, 1, \dots, m-1$, with the following commutation law: $[y_{1+iq}, y_{1+jq}] = 0$ for all i, j; $[x, y_{1+iq}] = y_{1+(i+1)q}$, where $(i+1)$ is taken modulo m. The ground field $\mathfrak{f}$ will be assumed algebraically closed of characteristic p, $(p, n) = 1$. Let Φ be the linear transformation of the vector space $\mathfrak{L}$:

$$\Phi(x) = \xi^q x, \quad \Phi(y_{1+iq}) = \xi^{1+iq} y_{1+iq}, \quad i = 0, 1, \dots, m-1;$$

ξ is a primitive n-th root of unity. It is easily verified that Φ is a regular automorphism of the algebra $\mathfrak{L}$ with period n, and that the algebra $\mathfrak{L}$ is solvable but nonnilpotent.

Second, a nilpotent Lie algebra with a regular automorphism of arbitrary period n may have an arbitrarily large nilpotency class. Let the ground field $\mathfrak{k}$ and the number n be the same as in the preceding example. Suppose further that the algebra $\mathfrak{L}$ is generated by the elements x, y^s_{1+iq}, $i = 0, 1, \dots, m-1$; $s = 1, 2, \dots, t$, where t is an arbitrary integer. Form⋯

commuting algebras $\mathfrak{L}$ commute with one another according to the following rules: $[y^s_{1+iq}, y^r_{1+jq}] = 0$ for all indices i, j, r, s; $[x, y^s_{1+iq}] = y^{s+1}_{1+(i+1)q}$, $(i+1)$ is taken modulo m, and $[x, y^t_{1+iq}] = 0$, $i = 0, 1, \dots, m-1$. The algebra $\mathfrak{L}$ is nilpotent, and its nilpotency class is equal to t. It is not difficult to see that the following linear transformation Φ of the vector space $\mathfrak{L}$ will be a regular automorphism of the algebra $\mathfrak{L}$ with period n: $\Phi(x) = \xi^q x$, $\Phi(y^s_{1+iq}) = \xi^{1+iq} y^s_{1+iq}$; $i = 0, 1, \dots, m-1$; $s = 1, 2, \dots, t$; ξ is a primitive n-th root of 1. Since t can be chosen arbitrarily large, the nilpotency class of the algebra $\mathfrak{L}$ can also be arbitrarily large.

In the present note we consider only the case where the regular automorphism has finite period. The paper (^{3}) gives a certain approach to the study of finite-dimensional Lie algebras over a field of characteristic $p \geqslant 0$ with an automorphism satisfying an arbitrary equation $f(x) = 0$, none of whose roots is equal to 1.

3. We shall prove the theorem formulated in item 1. Let there be an algebra $\mathfrak{L}$ with a regular automorphism Φ of period n. Without loss of generality one may assume that the ground field $\mathfrak{k}$ is algebraically closed. If p is the characteristic of the field $\mathfrak{k}$, then Φ^p is a regular automorphism. Suppose this is not so. Then the set of elements $x \in \mathfrak{L}$ such that $\Phi^p(x) = x$ is a nonzero Φ-invariant subalgebra $\mathfrak{L}_1$ of the algebra $\mathfrak{L}$. Consequently, the algebra $\mathfrak{L}_1$ over the field of characteristic p admits a regular automorphism Φ of period p, which is impossible (^{1}). Thus, if the algebra $\mathfrak{L}$ admits a regular automorphism of finite period, then it admits a regular automorphism whose period is not divisible by the characteristic of the ground field. We shall therefore assume from the outset that $(n, p) = 1$. The latter means that all roots of the minimal equation for Φ are distinct.

The algebra $\mathfrak{L}$, considered as a vector space over the field $\mathfrak{k}$, is represented in the form

$$\mathfrak{L} = \sum_{i=1}^{n-1} \mathfrak{L}_i,$$

where $\mathfrak{L}_i$ is the root subspace corresponding to the root ξ^i of the minimal equation for Φ; here ξ is a primitive root of degree n of 1. The subspace $\mathfrak{L}_0 = \{0\}$ by virtue of the regularity of Φ. Since Φ is an automorphism, $[\mathfrak{L}_i\mathfrak{L}_j] \subseteq \mathfrak{L}_{i+j}$, $(i+j)$ is taken modulo n. We agree to regard residues i modulo n as positive, $1 \leqslant i \leqslant n-1$. The set of residues is ordered in the natural way:

$$1 < 2 < \cdots < n-1.$$

Let $\mathfrak{L} = \mathfrak{L}^{(0)} \supseteq \mathfrak{L}^{(1)} \supseteq \cdots \supseteq \mathfrak{L}^{(k)} \supseteq \cdots$ be the derived series of the algebra $\mathfrak{L}$. Since all $\mathfrak{L}^{(s)}$, $s = 1, 2, \ldots, k, \ldots$, are Φ-invariant, they are also representable in the form

$$\mathfrak{L}^{(s)} = \sum_{i=1}^{n-1} \mathfrak{L}_i^{(s)},$$

where $\mathfrak{L}_i^{(s)}$ is the root subspace of the vector space $\mathfrak{L}^{(s)}$ belonging to the root ξ^i. Obviously,

$$\mathfrak{L}_i^{(s)} = \mathfrak{L}_i \cap \mathfrak{L}^{(s)}$$

and $\mathfrak{L}_i^{(s)} \subseteq \mathfrak{L}_i^{(t)}$ for $t \leqslant s$. Elements of $\mathfrak{L}_i^{(s)}$ will be denoted by $x_i^{(s)}$ (for simplicity one symbol is used, although in the expressions below different elements of $\mathfrak{L}_i^{(s)}$ may occur). We shall further denote by L_k the subalgebra generated by the subspaces $\mathfrak{L}_i$, $i \geqslant k$. Since $\mathfrak{L}^{(1)} = [\mathfrak{L}, \mathfrak{L}]$, the subspace $\mathfrak{L}_1^{(1)}$ consists of elements

$$x_1^{(1)} = \sum_{i=2}^{\left[\frac{n+1}{2}\right]} [x_i, x_{n+1-i}], \quad x_i \in \mathfrak{L}_i$$

and their linear combinations. This means that $x_1^{(1)} \in L_2$, i.e. $\mathfrak{L}_1^{(1)} \subseteq L_2$. Suppose, by induction, that the subspaces

$$\mathfrak{L}_i^{(2^k-1)} \subseteq L_k, \quad i \leqslant k-1.$$

We shall then show that

$$\mathfrak{L}_i^{(2^k-1)} \subseteq L_{k+1}, \quad i \leqslant k.$$

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indeed, as above, the subspace $\Omega_k^{(2^k-1)}$ consists of elements

$$x_k^{(2^k-1)} = \sum_{j=1}^{\left[\frac{k}{2}\right]} \left[x_j^{(2^{k-1}-1)}, x_{k-j}^{(2^{k-1}-1)}\right] + \sum_{j=k+1}^{\left[\frac{n+k}{2}\right]} \left[x_j^{(2^{k-1}-1)}, x_{n+k-j}^{(2^{k-1}-1)}\right] \qquad (*)$$

and their linear combinations. Here and below in the proof the following elementary fact is used: if $i_0 \equiv i_1 + i_2 \pmod{n}$, where $0 < i_j < n$, $j = 0, 1, 2$, then either both summands on the right-hand side of the congruence are less than i_0, or both are greater than i_0. Indeed, if $i_0 > i_1$, then $i_2 = i_0 - i_1 < i_0$; if, however, $i_0 < i_1$, then $i_2 = n + i_0 - i_1 = (n - i_1) + i_0 > i_0$. From this remark it follows that the second sum on the right-hand side of $(*)$ is contained in L_{k+1}.

By the induction hypothesis, $\Omega_i^{(2^{k-1}-1)} \subset L_k$ for $i \le k-1$. Consequently, an arbitrary element

$$x_{k-j}^{(2^{k-1}-1)} = \sum [x_{i_1}, x_{i_2}, \dots, x_{i_m}], \quad \text{where } 1 \le j \le \left[\frac{k}{2}\right], \ x_{i_s} \in \Omega_{i_s}, \ i_s \ge k,$$

where under the summation sign there stands a left-normed product. Substitute the value of $x_{k-j}^{(2^{k-1}-1)}$ in $(*)$ and consider an arbitrary term of the resulting sum:

$$\left[x_j^{(2^{k-1}-1)} [x_{i_1}, \dots, x_{i_m}]\right] = \sum \alpha_\pi \left[x_j^{(2^{k-1}-1)} x_{\pi i_1}, \dots, x_{\pi i_m}\right],$$

where π is some permutation of the symbols $i_1, i_2, \dots, i_m$, and under the summation sign there stand left-normed products. Two cases are possible: $\pi i_m = k$ and $\pi i_m > k$. If $\pi i_m = k$, then $j + \sum_{s=1}^{m-1} \pi i_s = 0$, and, consequently,

$$[x_j^{(2^{k-1}-1)}, x_{\pi i_1}, \dots, x_{\pi i_m}] = 0.$$

If, however, $\pi i_m > k$, then $j + \sum_{s=1}^{m-1} \pi i_s > k$. Then

$$[x^{(2^{k-1}-1)}, x_{\pi i_1}, \dots, x_{\pi i_{m-1}}] = y_t \in \Omega_t, \quad t = j + \sum_{s=1}^{m-1} \pi i_s > k;$$

hence

$$[x^{(2^{k-1}-1)} x_{\pi i_1}, \dots, x_{\pi i_m}] = [y_t, x_{\pi i_m}] \in L_{k+1}.$$

Thus, the first sum on the right-hand side of $(*)$ belongs to L_{k+1}, i.e. $\Omega_k^{(2^k-1)} \subset L_{k+1}$. The preceding arguments are applicable to any algebra with a regular automorphism of period n, in particular to the algebra $\mathfrak{N} = \Omega^{(2^k-1)}$. By the induction hypothesis, $\mathfrak{N}_i^{(2^{k-1}-1)} \subset N_k$, $i \le k-1$. The subalgebra N_k in $\mathfrak{N}$ corresponds to the subalgebra L_k in Ω. But, by what has been proved, $\Omega_k^{(2^k-1)} = \mathfrak{N}_k \subset L_{k+1}$, and since $N_{k+1} \subset L_{k+1}$, it follows that $\Omega_i^{(2^k-1)} = \mathfrak{N}_i^{(2^{k-1}-1)} \subset L_{k+1}$, $i \le k-1$. Since $\Omega_k^{(2^k-1)} \subset \Omega_k^{(2^{k-1})}$, we have $\Omega_k^{(2^n-1)} \subset L_{k+1}$, and the induction hypothesis is justified. For $k = n-2$, $\Omega_i^{(2^{n-2}-1)} \subset L_{n-1}$, $i \leq n-2$, i.e. $\Omega^{(2^{n-2}-1)} \subset L_{n-1}$. From this it is not difficult to obtain that $\Omega_{n-1}^{(2^{n-2})} = \{0\}$ and $\Omega^{(2^n-1)} = \{0\}$, as was required to be proved.

In the case when the period of the regular automorphism is equal to a prime number q, Higman' s construction (1), p. 327, makes it possible to apply the technique of the proof of the preceding theorem and Theorem B (3) to an arbitrary Lie ring admitting the indicated automorphism. Hence, as was already said in item 1, our corollary is obtained.

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Note: Figure translations are in progress. See original paper for figures.

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