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Abstract

Full Text

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MEASURES IN FUNCTION SPACE AND DIFFERENCE SCHEMES

(Presented by Academician P. S. Aleksandrov, 10 X 1962)

MATHEMATICS

1°. Consider the Cauchy problem for the equation

$$\partial u / \partial t = Lu \quad \text{with the initial condition} \quad u(0, x) = \varphi(x), \quad (1)$$

where L is a linear differential operator, and $\varphi(x)$ is a continuous bounded function. We shall assume that problem (1) has a unique solution $u(t, x)$, continuous and bounded in the strip $[0 \leq t \leq T, -\infty < x < \infty]$. Suppose, further, that the fundamental solution $G(t_0, x_0, t, x)$ of problem (1) satisfies the following conditions: 1) for fixed t_0, x_0 , and t ($t > t_0$), it is a continuous bounded nonnegative function of x ; 2)

$$\int_{-\infty}^{\infty} G(t_0, x_0, t, x) dx \leq 1 + c(t - t_0),$$

where c is a positive constant; 3)

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{|x-x_0| \geq \delta} G(t_0, x_0, t_0 + \Delta t, x) dx = 0$$

for every $\delta > 0$. An example of an equation (1) satisfying the conditions listed is the diffusion equation $\partial u / \partial t = a^2 \partial^2 u / \partial x^2$.

2°. Let

$$\Omega_{t_0}^{t,x} = \{\omega\} \quad (2)$$

be the set of continuous functions $\omega = \xi(\tau)$, defined on the interval $[t_0, t]$ and satisfying the condition $\xi(t) = x$ (i.e., their value at the right endpoint of the interval is fixed). Define on $\Omega_{t_0}^{t,x}$ a measure $\mu_{t_0}^{t,x}$ as follows. Let

$$A_x = \{\omega : a_i \leq \xi(t_i) \leq b_i \ (i = 0, 1, \dots, n-1), \ \xi(t) = x\}, \quad (3)$$

where $t_0 < t_1 < \dots < t_{n-1} < t$; $-\infty \leq a_i < b_i \leq \infty$. (Obviously, $A_n \subset \Omega_{t_0}^{t,x}$.)
Put

$$\mu_{t_0}^{t,x}(A_n) = \int_{a_0}^{b_0} \dots \int_{a_{n-1}}^{b_{n-1}} G(t_0, x_0, t_1, x_1) \dots G(t_{n-1}, x_{n-1}, t, x) dx_0 \dots dx_{n-1}. \quad (4)$$

The collection of sets of the form (3)–“quasi-intervals”–forms a semiring. One can show that the measure constructed is σ -additive on this semiring and, consequently, can be extended to a σ -additive measure on the σ -algebra generated by this semiring. This measure we shall denote by $\mu_{t_0}^{t,x}$.

In an analogous way one can introduce the measures μ_{t_0, x_0}^t (here functions are considered that have a fixed value x_0 at the point t_0) and $\mu_{t_0, x_0}^{t,x}$ (the values at both endpoints of the interval are fixed). (See also (1), p. 82, the definition of the conditional Wiener measure.) The measure $\mu_{t_0}^{t,x}$ will be denoted simply by μ .

3°. The solution of the Cauchy problem (1) is written with the aid of the fundamental solution in the following form:

$$u(t, x) = \int_{-\infty}^{\infty} \varphi(x_0) G(0, x_0, t, x) dx_0. \quad (5)$$

This integral can be represented as an integral with respect to the measure μ of the functional defined on $\Omega = \Omega_0^{t,x}$ and equal to $F(\omega) = \varphi[\xi(0)]$, i.e.

$$u(t, x) = \int_{\Omega} F(\omega) d\mu. \quad (6)$$

4°. By a one-layer difference scheme, or, more briefly, a difference scheme, we shall mean a sequence of equations of the form

$$u_{i+1,k}^m = \sum_{j=-\infty}^{\infty} a_{ijk}^m u_{ij}^m \quad \text{with the initial condition} \quad u_{0k}^m = \varphi_k^m, \quad (7)$$

where $u_{ik}^m = u^m(ih_m, kl_m)$; h_m and l_m are the steps in t and in x ; m takes the values $1, 2, \dots$, with $h_m \rightarrow 0$ and $l_m \rightarrow 0$ as $m \rightarrow \infty$. We shall assume the coefficients a_{ijk}^m to be nonnegative (of course, we do not exclude the case in which, in each sum of the form (7), all a_{ijk}^m , except for a finite number, are equal to zero).

The continuous function $u^m(t, x)$, coinciding with u_{ik}^m at the mesh points and linear in t for $x = \text{const}$ and in x for $t = \text{const}$ in each rectangle $[ih_m \leq t \leq (i+1)h_m, kl_m \leq x \leq (k+1)l_m]$, will be called a solution of the difference

scheme (7). We shall say that the difference scheme (7) is stable with respect to problem (1) if, for any initial function $\varphi(x)$, with $\varphi_k^m = \varphi(kl_m)$, the equality

$$\lim_{m \rightarrow \infty} u^m(t, x) = u(t, x)$$

holds, meaning convergence of the solution of the difference scheme to the solution of problem (1). If this convergence is uniform in the strip $[0 \leq t \leq T, -\infty < x < \infty]$, then scheme (7) is called uniformly stable.

5°. Let us write the expression for the solution $u^m(t, x)$ at a fixed point (t, x) explicitly in terms of the coefficients and the initial condition of scheme (7). Let $t = (n + \theta_1)h_m$, $x = (q + \theta_2)l_m$, where n and q are integers, $\theta_1, \theta_2 \in (0, 1]$. In the space $\Omega = \Omega_0^{t,x}$, introduce the subset $\widehat{\Omega}_m$ consisting of polygonal lines $\gamma(\tau)$ with vertices at the points $\tau = ih_m$ ($i = 0, 1, \dots, n$), $\xi = kl_m$ ($k = 0, \pm 1, \pm 2, \dots$). Denote $k_i = k(i) = \gamma(ih_m)/l_m$ ($k(i)$ is an integer-valued function). An elementary calculation shows that

$$u^m(t, x) = \sum_{\gamma(\tau) \subset \widehat{\Omega}_m} \varphi_{k_0}^m p_m[\gamma(\tau)], \quad (8)$$

where

$$p_m[\gamma(\tau)] = \alpha_{t, k_n, x}^m \prod_{i=0}^{n-1} a_{i, k_i, k_{i+1}}^m,$$

$$\alpha_{t, k, x}^m = \begin{cases} (1 - \theta_1)(1 - \theta_2) + \theta_1(1 - \theta_2)a_{nq}^m + \theta_1\theta_2 a_{nq, q+1}^m, & \text{if } k = q, \\ (1 - \theta_1)\theta_2 + \theta_1(1 - \theta_2)a_{n, q+1, q}^m + \theta_1\theta_2 a_{n, q+1, q+1}^m, & \text{if } k = q + 1, \\ \theta_1(1 - \theta_2)a_{nkq}^m + \theta_1\theta_2 a_{nk, q+1}^m, & \text{for the remaining } k. \end{cases}$$

(Note that when $\theta_1 = \theta_2 = 1$, $\alpha_{t, k, x}^m = a_{n, k, q+1}^m$.)

Introduce in the space Ω a sequence of measures $\mu_1, \dots, \mu_m, \dots$ as follows: for any $A \subset \Omega$,

$$\mu_m(A) = \sum_{\gamma(\tau) \in A} p_m[\gamma(\tau)]. \quad (9)$$

Then (8) can be rewritten in the form

$$u^m(t, x) = \int_{\Omega} F(\omega) d\mu_m, \quad (10)$$

where $F(\omega) = \varphi[\xi(0)]$. Thus, to each difference scheme there corresponds a sequence of measures μ_m , concentrated on the polygonal lines from $\widehat{\Omega}_m \subset \Omega$.

We shall consider the connection that exists between the stability of a difference scheme and the convergence (in one sense or another) of the sequence of measures (9) to the measure corresponding to the differential equation (1).

6°. We first formulate some definitions concerning sequences of measures in a function space.

Let $C(\Omega)$ be the set of continuous bounded functionals defined on the space $\Omega = \Omega_0^{t,x}$. Further, let N be some subset of $C(\Omega)$. We shall say that a sequence of measures μ_m , defined on Ω , converges with respect to N , or simply N -converges, to the measure μ , if

$$\int_{\Omega} f(\omega) d\mu_m \rightarrow \int_{\Omega} f(\omega) d\mu \quad (11)$$

for every $f(\omega) \in N$. If, in particular, N coincides with all of $C(\Omega)$, then (11) means simply weak convergence of the sequence $\{\mu_m\}$ to the measure μ . It is easy to verify the following assertion: if a bounded sequence of measures converges with respect to N , then it also converges with respect to $[N]$, where $[N]$ is the linear hull, closed in the sense of the uniform topology, of the set N .

Let $A(\Omega)$ be the set of functionals $f(\omega) \in C(\Omega)$ whose values depend only on $\xi_0 = \xi(0)$, the initial value of the function $\xi(\tau)$, and let $B(\Omega)$ be the set of functionals with values depending on a finite number of $\xi_i = \xi(t_i)$, $i = 0, 1, \dots, n-1$, where n is any integer. We denote the closure $[B(\Omega)]$ by $T(\Omega)$. Below, in addition to weak convergence, we shall deal with A -convergence and T -convergence. In view of formulas (6) and (10), stability of the difference scheme is equivalent to A -convergence of the sequence of measures (9).

A family M of measures is called N -compact if from every sequence of its elements one can extract an N -convergent subsequence. In particular, a C -compact family will be called simply compact.

The following necessary and sufficient condition for compactness of a family of measures M defined on Ω is known (see, for example, (2)):

- 1) $\sup_{\mu \in M} \mu(\Omega) < \infty$;
- 2) for every $\varepsilon > 0$ there exists a compact set K_ε such that $\mu(\Omega - K_\varepsilon) < \varepsilon$ for all $\mu \in M$. In (2) one can find some simple conditions sufficient for compactness.

A difference scheme for which the sequence (9) is compact will be called a compact difference scheme.

7°. In view of the inclusions $A(\Omega) \subset T(\Omega) \subset C(\Omega)$, it is obvious that weak convergence of the sequence (9) implies its T -convergence, and T -convergence, in turn, implies stability of the difference scheme. The converse connection

between stability of a difference scheme and convergence of the sequence (9) is established by the following theorems.

Theorem 1. *If a difference scheme is uniformly stable, then the corresponding sequence of measures T -converges.*

Theorem 2. *For weak convergence of the sequence of measures (9), it is necessary and sufficient that this sequence T -converge and be compact.*

Remark. It is not difficult to construct an example of a T -convergent sequence which is not weakly convergent, i.e., a T -convergent sequence may be compact (just as a compact sequence may fail to be T -convergent).

From Theorems 1 and 2 it follows that

Theorem 3. *If a difference scheme is uniformly stable, then the corresponding sequence of measures converges weakly if and only if the difference scheme is compact.*

8°. Let us note some possible applications of the results obtained.

1. Approximate computation of continual integrals. Consider the integral

$$I = \int_{\Omega} f(\omega) d\mu \quad (12)$$

of a continuous bounded functional $f(\omega)$ with respect to a measure μ corresponding to some equation (1). Take a compact uniformly stable difference scheme corresponding to this equation, and let $\{\mu_m\}$ be the sequence of measures corresponding to this scheme. The integral with respect to μ_m , which in fact is a sum of no more than a countable number of terms, for sufficiently large m will give, in view of Theorem 3, the value of the integral (12) with any prescribed accuracy.

2. From the stability conditions for difference schemes, formulated in terms of relations between the steps of the scheme h and l , one can obtain certain information about the set on which the measure corresponding to the differential equation is concentrated. For example, for the diffusion equation

$$\partial u / \partial t = a^2 \partial^2 u / \partial x^2 \quad (13)$$

the difference scheme $(u_{i,j+1} - u_{i,j})/h = a^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})/l^2$, or, equivalently,

$$u_{i,j+1} = a^2 \frac{h}{l^2} u_{i-1,j} + \left(1 - 2a^2 \frac{h}{l^2}\right) u_{i,j} + a^2 \frac{h}{l^2} u_{i+1,j} \quad (14)$$

is stable when

$$\frac{1}{2a^2} \frac{l^2}{h} = \frac{1}{2a^2} \frac{(\Delta x)^2}{\Delta t} > 1$$

(see, for example, (3)).

It can be verified that under this condition the scheme (14) is uniformly stable and compact. As follows from Theorem 3, the Wiener measure corresponding to equation (13) is a weak limit of the sequence of measures corresponding to scheme (14). Hence it is easy to derive the (known) result that, for any $\varepsilon > 0$, the Wiener measure of the set of functions satisfying a Hölder condition of order $1/2 - \varepsilon$ is equal to 1.

3. Theorem 3 can be applied in resolving the question of whether a difference scheme that is stable for the Cauchy problem will remain stable in solving other problems—for example, a problem with boundary conditions. An important property of the difference scheme in this case turns out to be its compactness.

9°. It is possible that the considerations set forth above can also be applied in the case when for equation (1) there is no corresponding σ -additive measure, but there exist stable single-layer difference (not sign-constant) schemes. These schemes make it possible to construct a sequence ν_m of additive set functions concentrated on polygonal lines; however, such a sequence may no longer be C -convergent. An example of such an equation is equation (13) with complex a^2 . The basic question here is to find that (as large as possible) set of functionals from $C(\Omega)$ with respect to which the sequence ν_m converges.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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