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Abstract

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ON THE QUESTION OF INTEGRAL INEQUALITIES FOR VOLTERRA-TYPE EQUATIONS WITH DELAYED ARGUMENT

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In the present paper we consider some properties of solutions of integral equations with delayed argument

$$x(t) = \int_a^t F(t, s, x(s - g_1(s)), \dots, x(s - g_n(s)))\, ds + f(t), \tag{1}$$

where:

A. $F(t, s, x_1, \dots, x_n)$ is continuous in the domain $a \leqslant s \leqslant t < b$, $|x_k| < C$ $(k = 1, 2, \dots, n)$, and

$$F(t, s, x_1^{(2)}, \dots, x_n^{(2)}) \geqslant F(t, s, x_1^{(1)}, \dots, x_n^{(1)}),$$

if $x_1^{(2)} \geqslant x_1^{(1)}, \dots,\ x_n^{(2)} \geqslant x_n^{(1)};\ |x_k^{(i)}| < C\ (i = 1, 2;\ k = 1, 2, \dots, n)$;
$F(t, s, x_1, \dots, x_n) \equiv 0$ for $s > t$.

B. Continuous delays $g_k(t) \geqslant 0$ $(k = 1, 2, \dots, n)$ are defined for $t \in [a, b)$. The functions $g_k(t)$ $(k = 1, 2, \dots, n)$ determine the initial set E_0, consisting of the point a and of the values of the difference $t - g_k(t) \leqslant a$ for $a \leqslant t < b$ and $k = 1, 2, \dots, n$.

C. The function $f(t)$ is continuous for $t \in [a, b) \cup E_0$ and $|f(t)| < C$ for $t \in E_0$.

Let us note that, by property A, the function $f(t)$ determines the solution of equation (1) on the initial set. Such equations arise in the study of problems of pulse engineering (see, for example, ([1])).

1. We first consider the linear equation

$$x(t) = \int_a^t \sum_{m=1}^{n} K_m(t, s)\, x(s - g_m(s))\, ds + f(t). \tag{2}$$

By the classical method of successive approximations (see, for example, ([2,3])) one can easily prove that equation (2) has a unique continuous solution. This solution is determined by the formula

$$x(t) = \int_a^t \sum_{m=1}^{n} R_m(t,s)\, f(s - g_m(s))\, ds + f(t),$$

where $R_m(t,s)$ $(m = 1, 2, \dots, n)$ are the resolvent kernels of equation (2), determined by means of a series composed of iterated kernels

$$R_m(t,s) = \sum_{k=1}^{\infty} K_m^{(k)}(t,s) \quad (m = 1, 2, \dots, n),$$

where $K_m^{(1)}(t,s) = K_m(t,s)$ $(m = 1, 2, \dots, n)$, and each of the subsequent kernels is determined by the recurrence relation

$$K_m^{(k)}(t,s) = \int_s^t \sum_{i=1}^{n} K_i(t,z)\, K_m^{(k-1)}(z - g_i(z), s)\, dz.$$

The resolvent kernels satisfy the integral equations

$$R_m(t,s) = K_m(t,s) + \int_s^t \sum_{i=1}^{n} K_i(t,z) R_m(z - g_i(z), s)\, dz =$$

$$= K_m(t,s) + \int_s^t \sum_{i=1}^{n} R_i(t,z) K_m(z - g_i(z), s)\, dz \quad (m = 1, 2, \dots, n).$$

Of great interest for the equations under consideration are theorems on integral inequalities (see, for example, (4-6)). We present several theorems of this type.

Theorem 1. Suppose that for equation (2) and the equation

$$y(t) = \int_a^t \sum_{m=1}^{n} G_m(t,s) y(s - h_m(s))\, ds + \varphi(t) \tag{3}$$

the following conditions are satisfied:

1) $G_m(t,s) \geq K_m(t,s) \geq 0;\quad K_m(t - h_m(t), s) \geq K_m(t - g_m(t), s)$

$$(m = 1, 2, \dots, n);$$

2) $\varphi(t) \geq f(t) \geq 0;\quad f(t - h_m(t)) \geq f(t - g_m(t)) \quad (m = 1, 2, \dots, n)$.

Then $y(t) \geq x(t)$, where $y(t)$ and $x(t)$ are the solutions of equations (3) and (2), respectively.

From this theorem, as a special case, follows the theorem on integral inequalities for linear Volterra integral equations of the second kind.

2. We formulate a comparison theorem for the solutions of equation (1) and the equation

$$y(t) = \int_a^t G(t, s, y(s - h_1(s)), \dots, y(s - h_n(s)))\, ds + a(t) \tag{4}$$

under the assumption that the solutions $y(t)$ and $x(t)$ of equations (4) and (1) exist and are unique for $t \in [a, b) \cup E_0$.

Theorem 2. Suppose the following conditions are satisfied:

$$\begin{aligned} 1) \quad & \int_{t_0}^{t-h_k(t)} G(t - h_k(t), s, x(s - g_1(s)), \dots, x(s - g_n(s)))\, ds \\ & - \int_{t_0}^{t-g_k(t)} F(t - g_k(t), s, x(s - g_1(s)), \dots, x(s - g_n(s)))\, ds \\ & + [\varphi(t - h_k(t)) - f(t - g_k(t))] \geq 0 \end{aligned}$$

for all continuous $x(t)$ such that $|x(t)| < C,\ t_0 \in [a, b),\ k = 1, 2, \dots, n$.

2) $G(t, s, x(s-g_1(s)), \dots, x(s-g_n(s))) \geq F(t, s, x(s-g_1(s)), \dots, x(s-g_n(s)))$ for all continuous $x(t)$ such that $|x(t)| < C$.

3) $\varphi(t) \geq f(t)$.

Then, for $t \in [a, b) \cup E_0$, the inequality $y(t) \geq x(t)$ holds, where $y(t)$ and $x(t)$ are the solutions of equations (4) and (1), respectively.

Proof. Let the solution of the integral equation (1) exist and be unique for $t \in [a, b) \cup E_0$. If the function $z(t)$ ($|z(t)| < C$ for $t \in [a, b) \cup E_0$) satisfies the integral inequality

$$z(t) \geq \int_a^t F(t, s, z(s - g_1(s)), \dots, z(s - g_n(s)))\, ds + f(t),$$

then for $t \in [a, b) \cup E_0$ the inequality $z(t) \geq x(t)$ holds, where $x(t)$ is the solution of equation (1).

As is known (see, for example, (5, 7-11)), comparison theorems make it possible to obtain nonlocal existence theorems, and theorems on uniqueness, stability, and the asymptotic behavior of solutions. As an example, let us consider the following uniqueness theorems.

For equation (1), denote by P the set of values t^* on the interval $a \leqslant t < b$ for which the inequality $t - g_k(t) \leqslant t^*$ holds for $t \in [t^*, t^* + \varepsilon)$, where $\varepsilon > 0$ and $k = 1, 2, \dots, n$.

Theorem 3. *Suppose that, for each point $t_0 \in P$, in the domain $t_0 \leqslant s \leqslant t \leqslant t_0 + \varepsilon$, where $\varepsilon > 0$, the following conditions are satisfied:*

1) *$F(t, s, x_1, \dots, x_n)$, for arbitrary continuous $x^{(2)}(t)$ and $x^{(1)}(t)$ ($|x^{(2)}(t)| < C$; $|x^{(1)}(t)| < C$), satisfies the inequality*

$$\begin{aligned} &\big|F(t, s, x^{(2)}(s - g_1(s)), \dots, x^{(2)}(s - g_n(s))) \\ &\quad - F(t, s, x^{(1)}(s - g_1(s)), \dots, x^{(1)}(s - g_n(s)))\big| \\ &\leqslant G(t, s,\ |x^{(2)}(s - g_1(s)) - x^{(1)}(s - g_1(s))|, \dots \\ &\qquad\qquad \dots,\ |x^{(2)}(s - g_n(s)) - x^{(1)}(s - g_n(s))|). \end{aligned}$$

2) *The equation*

$$y(t) = \int_{t_0}^{t} G(t, s, y(s - g_1(s)), \dots, y(s - g_n(s)))\, ds$$

has the unique nonnegative solution $y(t) \equiv 0$.

Then the solution of equation (1) is unique.

This theorem shows that the classical scheme presented above for investigating questions in the qualitative theory of nonstationary problems is also applicable to equations with delay. Using this scheme, we give the following assertion, analogous to the theorem of A. D. Myshkis (12).

Theorem 4. *Suppose that, for each point $t_0 \in P$, in the domain $t_0 \leqslant s \leqslant t \leqslant t_0 + \varepsilon$, where $\varepsilon > 0$, the following conditions are satisfied:*

1) *There exist constant numbers $b_i > 0$, $\alpha_i > 1$ such that*

$$g_i(t) \geqslant (t - t_0) - b_i(t - t_0)^{\alpha_i}$$

for all $i = 1, 2, \dots, n$.

2) *$F(t, s, x_1, \dots, x_n)$, for arbitrary $x_i^{(2)}$ and $x_i^{(1)}$, where $|x_i^{(2)}| < C$; $|x_i^{(1)}| < C$; $i = 1, 2, \dots, n$, satisfies the inequality*

$$\big|F(t, s, x_1^{(2)}, \dots, x_n^{(2)}) - F(t, s, x_1^{(1)}, \dots, x_n^{(1)})\big| \leqslant K \sum_{i=1}^{n} |x_i^{(2)} - x_i^{(1)}|^{\frac{1}{\alpha_i}}.$$

Then the solution of equation (1) is unique.

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Note: Figure translations are in progress. See original paper for figures.