Reports of the Academy of Sciences of the USSR

1963

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1963. Vol. 153, No. 5

GEOPHYSICS

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APPLICATION OF STATISTICAL METHODS TO THE STUDY OF THE FINITE-DIFFERENCE STRUCTURE OF THE BALANCE EQUATION

(Presented by Academician A. A. Dorodnitsyn on 18 VI 1963)

Let us consider the errors of a difference approximation of the balance equation

$$l\nabla^2\psi + 2\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\,\partial y}\right)^2\right] + \frac{\partial\psi}{\partial x}\frac{\partial l}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial l}{\partial y} = \nabla^2\Phi. \tag{1}$$

Here l is twice the vertical component of the angular velocity of the Earth' s rotation, ψ is the stream function, Φ is the geopotential, x, y are horizontal coordinates, and ∇^2 is the two-dimensional Laplace operator in the plane x, y.

Equation (1) is widely used in the practice of numerical weather prediction ([2],[3]), for computing the wind field from the geopotential field. In doing so, one passes from differential operators to grid operators. On the basis of forecasting experience, the grid step h in most cases varies from 250 to 400 km, and reducing it, given the existing density of the aerological network and the accuracy of the initial data, seems inexpedient. Therefore, when choosing optimal methods of grid approximation, it is meaningless to be guided by methods of error estimation associated with expansion in powers of the small parameter h. It is more rational to rely on information about the statistical structure of meteorological fields ([5],[6]). Indeed, in problems of numerical weather prediction, where the same equations are integrated many times in order to find solutions under different initial conditions, it is entirely natural to regard the initial data as random fields possessing certain statistical regularities.

Let the differential equation under consideration have the form

$$\mathcal{L}f = 0, \tag{2}$$

Fig. 1

Figure 1: Fig. 1

where $\mathcal{L}$ is a differential operator.

For a given step and grid shape there remains the freedom to choose the angle ϑ between the coordinate axes and the axes determining the arrangement of the grid points (see Fig. 1); denote the corresponding finite-difference operator by $\mathcal{L}_{h,\vartheta}$. Then the grid equation corresponding to (2) may be written as

$$\mathcal{L}_{h\vartheta} f = \varepsilon, \tag{3}$$

where ε is the error of the difference approximation.

Fig. 1

The quantity ε can be determined at each grid point and is a random function of the coordinates, while the individual values of the function refer to different moments of time.

In order to be able to apply the theory of random fields in the investigation, let us randomize the position of the origin of coordinates. Then, far from the boundaries of the region, ε may be regarded as a statistically homogeneous field in the space of Cartesian coordinates x, y. Since the correlation function of the error field ε decreases rapidly with the distance between points*, then, for an approximate probabilistic estimate of the approximation error, the properties of the function may be extended to the infinite plane. Now let us carry out the spectral expansion of the error ε (see, for example, ([9])):

$$\varepsilon(x, y) = \int_{-\infty}^{\infty}\int e^{i\mathbf{k}\cdot\mathbf{r}}\, dZ_{\varepsilon}(k_1, k_2). \tag{4}$$

* Cf. also Table 1, showing that the spectral density of the error in the wavelength interval $\lambda \geqslant 8h$ (h is the grid step) becomes very small.

In the expression of the Fourier–Stieltjes integral (4), $\mathbf{k}\cdot\mathbf{r}$ is the scalar product of the wave vector $\mathbf{k}(k_1, k_2)$ and the radius vector of the point $\mathbf{r}(x, y)$; $dZ_{\xi}(k_1, k_2)$ are increments of the random function $Z_{\xi}(k_1, k_2)$.

Let us represent the solution of equation (1) in the form of the sum of the stream function of the basic motion (westward transport with some stationary velocity $U(y)$) and a deviation ψ'. We regard the latter as a homogeneous random function with mean value equal to zero, and represent it in the form of a Fourier –Stieltjes integral. We represent Φ' analogously. Then we obtain:

$$\psi = -\int U(y)\,dy + \int_{-\infty}^{\infty}\!\!\int e^{i\mathbf{k}\cdot\mathbf{r}}\,dZ_\psi(k_1,k_2),$$
$$\Phi = -l\int U(y)\,dy + \int_{-\infty}^{\infty}\!\!\int e^{i\mathbf{k}\cdot\mathbf{r}}\,dZ_\Phi(k_1,k_2). \tag{5}$$

In formulas (5) it is not indicated that the functions ψ and Φ depend parametrically on time t and on pressure p.

When replacing derivatives at the grid point O by finite differences, we use the 8 nearest grid points shown in Fig. 1.

Varying the angle ϑ, we shall try to make the mathematical expectation of the error $\bar{\varepsilon}$, hereafter called the systematic error, equal to zero. We then vary the expression for the difference Laplace operator Δ_h in order to reduce the error variance $\overline{\varepsilon^2}$.

To investigate the systematic error arising in passing to the difference form of equation (1), it is sufficient to consider the nonlinear terms of the equation. Indeed, under the condition $\psi'(x,y) = 0$, for any linear grid operator we have:

$$\overline{\mathcal{L}_{h,0}\psi'(x,y)} = 0.$$

The spectral expansion of the nonlinear terms of the differential balance equation is given in ([8]). Using it, it is not difficult to obtain an expression for the systematic error of the corresponding difference expression. For the case $\vartheta = 0$, when the second differences $\delta^2\psi/\delta x^2$ and $\delta^2\psi/\delta y^2$ are expressed through the values of the function at the points $0, 1, 3, 2, 4$, and the difference $\delta^2\psi/\delta x\,\delta y$ through the values of the function at the points $5, 6, 7, 8$, the systematic error takes the form

$$\bar{\varepsilon} = \frac{1}{h^4}\int_{-\infty}^{\infty}\!\!\int \left\{\sin^2\mu_1 h\,\sin^2\mu_2 h - 16\sin^2\frac{\mu_1 h}{2}\cdot\sin^2\frac{\mu_2 h}{2}\right\} dZ_\psi(\mu_1,\mu_2)\,dZ^*_\psi(\mu_1,\mu_2).$$

Let us next consider the case $\vartheta = \pi/4$, when the difference approximation for the nonlinear term is determined by the formula:

$$\frac{\delta^2\psi}{\delta x^2}\frac{\delta^2\psi}{\delta y^2} - \left(\frac{\delta^2\psi}{\delta x\,\delta y}\right)^2 =$$
$$= \frac{1}{4h^4}\left\{(\psi_5+\psi_7-2\psi_0)(\psi_6+\psi_8-2\varphi_0) - (\psi_1-\psi_2+\psi_3-\psi_4)^2\right\}. \tag{6}$$

In this case the quantity $\bar{\varepsilon}$ vanishes identically, which settles the question of the choice of the direction of the coordinate axes*.

Since the differential Laplace operator is invariant with respect to rotation of the coordinate axes, the question of choosing its finite-difference approximation is not completely solved by choosing the angle ϑ of rotation of the grid axes at the first stage of the investigation. We leave aside here the question of the difference approximation of the terms $\frac{\partial \psi}{\partial x}\frac{\partial l}{\partial x}$ and $\frac{\partial \psi}{\partial y}\frac{\partial l}{\partial y}$ as less essential.

When the 9 points indicated above are used, the expression of the operator Δ_h

* The advantages of a grid rotated by $\pi/4$ for calculating nonlinear terms were first indicated by B. Bolin ([10]

taking into account the symmetry properties has the form (1):

$$\Delta_h \psi = \frac{\alpha}{2h^2}(\psi_5 + \psi_6 + \psi_7 + \psi_8 - 4\psi_0) + \frac{(1-\alpha)}{h^2}(\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0). \quad (7)$$

Here α is a certain constant that we may choose in order to reduce the variance of the error.

Using the representation of the functions φ and Φ in the form (5), we obtain the spectral form of the differential balance equation:

$$(k_1^2 + k_2^2)\,[\,l\,dZ_\psi(k_1,k_2) - dZ_\Phi(k_1,k_2)\,] =$$

$$= \int_{-\infty}^{\infty}\!\!\int (\mu_1 k_2 - \mu_2 k_1)^2\, dZ_\psi(\mu_1,\mu_2)\, dZ_\psi^*(\mu_1 - k_1; \mu_2 - k_2). \quad (8)$$

The spectral form of the corresponding grid equation has the form:

$$\frac{4}{h^2}\left(\sin^2\frac{a_1+a_2}{2} + \sin^2\frac{a_1-a_2}{2} - 2\alpha\sin^2\frac{a_1+a_2}{2}\cdot\sin^2\frac{a_1-a_2}{2}\right)\times$$

$$\times[\,l\,dZ_\psi(k_1,k_2) - dZ_\Phi(k_1,k_2)\,] = \frac{4}{h^4}\int_{-\infty}^{\infty}\!\!\int [\sin b_1 \sin(b_2 - a_2) -$$

$$-\sin b_2 \sin(b_1 - a_1)]^2\, dZ_\psi(\mu_1,\mu_2)\, dZ_\psi^*(\mu_1 - k_1; \mu_2 - k_2) + \varepsilon(k_1,k_2), \quad (9)$$

where ε is the error of the difference approximation; a_1, a_2, b_1, b_2 are dimensionless variables: $a_1 = k_1 h\sqrt{2}/2$; $a_2 = k_2 h\sqrt{2}/2$; $b_1 = \mu_1 h\sqrt{2}/2$; $b_2 = \mu_2 h\sqrt{2}/2$.

Using formulas (8) and (9) and the expression for the spectral density of the product of two random functions ([8]), we obtain the expression for the spectral density of the error

$$S_\varepsilon(k_1,k_2)=\frac{32}{h^8}\int_{-\infty}^{\infty}\!\!\int\left\{\frac{2\left(\sin^2\frac{a_1+a_2}{2}+\sin^2\frac{a_1-a_2}{2}-2\alpha\sin^2\frac{a_1+a_2}{2}\sin^2\frac{a_1-a_2}{2}\right)(b_1a_2-b_2a_1)^2}{(a_1^2+a_2^2)}-\right.$$

$$\left.-[\sin b_1\sin(b_2-a_2)-\sin b_2\sin(b_1-a_1)]^2\right\}^2\times$$

$$\times S_\psi(\mu_1,\mu_2)\,S_\psi^*(k_1-\mu_1;\,k_2-\mu_2)\,d\mu_1 d\mu_2. \tag{10}$$

In specific calculations it was assumed that the spectral density of the stream-function field S_ψ is proportional to the spectral density of the geopotential field, estimated in ([7]):

$$S_\psi(k_1,k_2)=\frac{3\beta_0^3B_\psi(0)}{2\pi[\beta_0^2+k^2]^{5/2}}, \tag{11}$$

where $B_\psi(0)$ and β_0 are constant quantities, $\beta_0=10^{-6}$ m^{-1}.

Formula (1) is based on a small amount of experimental material, but apparently describes with sufficient accuracy the structure of the field in that interval of wave-vector values outside which the integrand in the right-hand side of (10) is relatively small.

The calculations of the spectral densities of the error were performed for the following values of the wave vector: $a_1=\pi/6,\ a_2=0$; $a_1=\pi/3,\ a_2=0$; $a_1=2\pi/3,\ a_2=0$; $a_1=\pi/6\sqrt{2},\ a_2=\pi/6\sqrt{2}$; $a_1=\pi/3\sqrt{2},\ a_2=\pi/3\sqrt{2}$; $a_1=2\pi/3\sqrt{2},\ a_2=2\pi/3\sqrt{2}$.

The calculations were carried out on the Ural-1 computer. In computing the double integrals, the integration domain was chosen on the basis of the requirement that the values of the integrands on the boundaries of the domain should not exceed 1% of their maximum values.

For each region of the spectrum, the value of the parameter $\alpha=\alpha_0$ was estimated at which the spectral density of the error in the given interval of wave-vector values becomes minimal. These values, as well as the values of the spectral density of the error for $\alpha=\alpha_0$, 0, and 1, are given-

Table 1

Values of the parameter α_0 and of the spectral density of the errors $S_\varepsilon(a,0)/B_\psi^2(0)\beta_0^6$ and

$$S_\varepsilon\left(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right)/B_\psi^2(0)\beta_0^6$$

a	λ/h	α_0	$S_\varepsilon(a,0)/B_\psi^2(0)\beta_0^6$, $\alpha=\alpha_0$	$S_\varepsilon(a,0)/B_\psi^2(0)\beta_0^6$, $\alpha=1$	$S_\varepsilon(a,0)/B_\psi^2(0)\beta_0^6$, $\alpha=0$	$\dfrac{S_\varepsilon\left(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right)}{\beta_\psi^2(0)\beta_0^6}$
$\pi/6$	8,4	1,77	0,0052	0,0061	0,0099	0,0123
$\pi/3$	4,2	1,14	0,0782	0,0791	0,1375	0,0684
$2\pi/3$	2,1	0,84	0,0160	0,0243	0,2574	0,0727

are given in Table 1. In the last column of the table are given the values of the spectral densities of the error for $a_1 = a_2$, when the choice of the parameter α becomes immaterial. It is seen that the errors increase as the wavelength decreases. From a comparison of the second and fourth columns of Table 2 it follows that replacing the optimal value of the parameter α by unity increases the error only insignificantly.

Table 2

Relative error of approximation of the grid balance equation

a	$\sqrt{\dfrac{S_\varepsilon(a,0)}{S^{(3)}}}$, $\alpha=\alpha_0$	$\sqrt{\dfrac{S_\varepsilon(a,0)}{S^{(3)}}}$, $\alpha=1$	$\sqrt{\dfrac{S_\varepsilon(a,0)}{S^{(3)}}}$, $\alpha=0$	$\sqrt{\dfrac{S_\varepsilon\left(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right)}{S^{(3)}(a)}}$
$\pi/6$	0,12	0,13	0,17	0,19
$\pi/3$	0,30	0,30	0,40	0,28
$2\pi/3$	0,11	0,13	0,43	0,23

In view of the fact that the idea of the quasi-solenoidal approximation consists in taking account of the nonlinear term discarded in the quasi-geostrophic approximation, it is natural to relate the calculated values of the spectral densities of the error to the quantity $S^{(3)}$, the spectral density of the nonlinear terms in the balance equation ([8]).

It follows from Table 2 that these ratios are not small, especially when the parameter α is equal to zero. These values somewhat exceed the relative error at values of the wave numbers $a_1 = a_2$, so that refinement of the difference scheme by varying the parameter α appears expedient. It makes it possible to use, to a greater extent, the advantages of the quasi-solenoidal forecast model over the cruder quasi-geostrophic one, than has been done in the works of a number of authors ([10–12]).

In conclusion we note that the proposed method may have a rather general significance in the analysis of many natural processes for which the statistical characteristics of the fields under study (structural or correlation functions) are known and which are described by equations of mathematical physics.

Main Geophysical Observatory
named after A. I. Voeikov

Received
14 VI 1963

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Note: Figure translations are in progress. See original paper for figures.