A CONSTRUCTIVE MAPPING OF THE SQUARE INTO ITSELF THAT MOVES EACH CONSTRUCTIVE POINT

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Abstract

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A CONSTRUCTIVE MAPPING OF THE SQUARE INTO ITSELF THAT MOVES EACH CONSTRUCTIVE POINT

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In the present communication we shall use the notions of a constructive operator from one constructive metric space into another, a continuous constructive operator, and a uniformly continuous constructive operator. Definitions of these notions may be found in [1]. Following [1], we denote by $\mathcal{E}_{2,2}$ the space of pairs of constructive real numbers*, i.e., words of the form $x\sigma y$, where x and y are constructive real numbers. In what follows, pairs of constructive real numbers will be called **points**.

A **lattice** will mean any word of the form $P\square x\Delta y$, where P is a system of pairwise disjoint rational segments (see [2]) and $x\Delta y$ is a segment containing the system P. We shall say that the point $x\sigma y$ belongs to the lattice $P\square x_1\Delta y_1$, and shall write $x\sigma y \in P\square x_1\Delta y_1$, if

$$(x \in P \;\&\; y \in x_1\Delta y_1) \quad \text{or}^{**} \quad (y \in P \;\&\; x \in x_1\Delta y_1).$$

The lattice $x\Delta y\square x\Delta y$ will be called a **square** with vertices at the points $x\sigma x$, $x\sigma y$, $y\sigma x$, $y\sigma y$ and will be denoted by xty. We shall say that the point $x_1\sigma y_1$ **lies on the boundary of the square** xty if $x_1\sigma y_1 \in xty$ and

$$x_1 = x, \quad \text{or } x_1 = y, \quad \text{or } y_1 = x, \quad \text{or } y_1 = y.$$

The inclusion relation for lattices is defined in the natural way and is denoted by the sign $\subseteq$.

Definition 1. A constructive operator F from $\mathcal{E}_{2,2}$ into $\mathcal{E}_{2,2}$ will be called a **retraction of the square** xty **onto its boundary** if the following conditions are satisfied:

1) whatever the point $x_1\sigma y_1$ belonging to the square xty, F is applicable to the point $x_1\sigma y_1$ and $F(x_1\sigma y_1)$ belongs to the boundary of the square xty;
2) whatever the point $x_1\sigma y_1$ belonging to the boundary of the square xty,

$$F(x_1\sigma y_1) = x_1\sigma y_1.$$

Definition 2. A constructive operator F from $\mathcal{E}_{2,2}$ into $\mathcal{E}_{2,2}$ will be called a **mapping of the square xty into itself without a fixed point** if, whatever the point $x_1\sigma y_1$ belonging to xty, F is applicable to the point $x_1\sigma y_1$ and

$$(F(x_1\sigma y_1) \in xty \ \&\ F(x_1\sigma y_1) \neq x_1\sigma y_1).$$

Theorem 1. *One can construct a continuous retraction of the square $-1t1$ onto its boundary.*

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* Here and below, by a constructive real number we mean a real FR-number (a real duplex) [1].
** Following [1], by the expression "A or B" we shall mean the formula $\neg\neg(A \vee B)$.

Theorem 2. One can construct a continuous mapping F of the square $-1\tau 1$ into itself, without a fixed point, such that for any point $x\sigma y$ belonging to the square $-1\tau 1$,

$$\rho(F(x\sigma y) \,\square\, x\sigma y) \geqslant {}^1/_8, \tag{1}$$

where ρ is the metric function in the space $\overline{\mathcal{E}}_{2,2}$.

Theorem 3. One can construct a uniformly continuous mapping of the square $-1\tau 1$ into itself, without a fixed point.

Let us outline the plan of proof of these theorems. First, algorithms $\mathfrak{A}$ and $\mathfrak{B}$ are constructed such that:

1) the algorithm $\mathfrak{A}$ transforms any natural number into a system of pairwise disjoint rational segments contained in the segment $-{}^1/_2\Delta{}^1/_2$ and consisting of an odd number of segments;

2) for any natural number n,

$$\mathfrak{A}(n+1) \,\square\, -1\Delta 1 \subseteq \mathfrak{A}(n) \,\square\, -1\Delta 1 \subseteq -1\tau 1;$$

3) the algorithm $\mathfrak{B}$ transforms any constructive real number x into such a natural number that

$$\neg(x \in \mathfrak{A}(\mathfrak{B}(x))).$$

The construction of the algorithms $\mathfrak{A}$ and $\mathfrak{B}$ differs only slightly from the construction of analogous algorithms in Theorems 4.1 and 4.2 of [2].

The most difficult part of the proof of the theorem formulated above is the construction of an algorithm transforming any natural number n into such a uniformly continuous operator Φ_n from $\overline{\mathcal{E}}_{2,2}$ into $\overline{\mathcal{E}}_{2,2}$ that, whatever the point $x\sigma y$ may be:

1) if $x\sigma y$ belongs to $-1\tau 1$, then Φ_n is applicable to $x\sigma y$ and $\Phi_n(x\sigma y) \in -1\tau 1$.
2) if $x\sigma y$ belongs to the boundary of the square $-1\tau 1$, then

$$\Phi_n(x\sigma y) = x\sigma y;$$

3) if $x\sigma y$ does not belong to the lattice $\mathfrak{A}(n)\,\square-1\Delta 1$, then $\Phi_n(x\sigma y)$ belongs to the boundary of the square $-1\tau 1$ and

$$\Phi_{n+1}(x\sigma y) = \Phi_{n+2}(x\sigma y).$$

After the algorithms characterized above have been constructed, constructive operators G and F_0 are constructed so that, for any point $x\sigma y$,

$$G(x\sigma y) \simeq \Phi_{\max(\mathfrak{B}(x),\,\mathfrak{B}(y))+1}(x\sigma y),$$

$$F_0(x\sigma y) \simeq I(x\sigma y) - 2^{-3}\cdot G(x\sigma y),$$

where I is the identity constructive operator from $\overline{\mathcal{E}}_{2,2}$ into $\overline{\mathcal{E}}_{2,2}$.

The continuity of the operators G and F_0 follows from the main theorem of [3]. It is easy to see that the operator G is a retraction of the square $-1\tau 1$ onto its boundary, while the operator F_0 is a mapping of the square $-1\tau 1$ into itself, without a fixed point, satisfying condition (1).

For the proof of Theorem 3, an algorithm is constructed which transforms any natural number n into such a uniformly continuous operator φ_n from $\overline{\mathcal{E}}_{2,2}$ into the space of constructive real numbers that, for any point $x\sigma y$ belonging to $-1\tau 1$:

1) the operator φ_n is applicable to the point $x\sigma y$, and $0 \leqslant \varphi_n(x\sigma y) \leqslant 1$;
2) $\varphi_{n+1}(x\sigma y) = 0 \equiv x\sigma y \in \mathfrak{A}(n)\,\square-1\Delta 1$.

It is easy to see that, for any point of the square $-1\tau 1$, the identity algorithm h is a regulator of convergence into itself of the sequence g

such that, for every n,

$$g(n) \simeq \sum_{i=1}^{n} 2^{-i}\varphi_i(x\sigma y)\cdot\Phi_i(x\sigma y).$$

The limit of the sequence g, constructed on the basis of the regulator of convergence in itself h, will be denoted by $W(x\sigma y)$. We now construct a constructive operator H such that, for any point $x\sigma y$,

$$H(x\sigma y) \simeq I(x\sigma y) - 2^{-4} \cdot W(x\sigma y).$$

The operator H is a uniformly continuous mapping of the square $-1\tau 1$ into itself without a fixed point.

We shall say that a mapping F of the square $x_1 \tau y_1$ into itself is a **pseudouniformly continuous mapping** if

$$\forall n \neg\neg\exists m \, \forall xyzu \, ((x\sigma y, z\sigma u \in x_1 \tau y_1 \& \rho(x\sigma y \square z\sigma u) <$$

$$< 2^{-m}) \supset \rho(F(x\sigma y) \square F(z\sigma u)) < 2^{-n}).$$

This definition extends naturally to the case of an arbitrary constructive metric space.

The following theorem may be regarded as a constructive analogue of Brouwer' s fixed-point theorem.

Theorem 4. *Whatever constructive pseudouniformly continuous mapping F of the square $-1\tau 1$ into itself may be, for every l there is potentially realizable a rational point $a\sigma b$, belonging to $-1\tau 1$, such that*

$$\rho(F(a\sigma b) \square a\sigma b) < 2^{-l}.$$

In the proof of this theorem one uses the fact that, for a piecewise-linear (simplicial) mapping of a square into itself, under which all vertices of both triangulations of the square are rational points, Brouwer' s theorem is transferred to constructive mathematics verbatim.

Let $\overline{\mathcal{E}}_{3,2}$ denote the space of triples of constructive real numbers. We shall say that a point $x\sigma y\sigma z$ of this space belongs to the **unit two-dimensional sphere** if $x^2 + y^2 + z^2 = 1$. A constructive operator F from $\overline{\mathcal{E}}_{3,2}$ into $\overline{\mathcal{E}}_{3,2}$ will be called a **vector field** defined on the unit two-dimensional sphere if it is applicable to all points belonging to the unit two-dimensional sphere.

From Theorem 3 there follow the following theorems:

Theorem 5. *One can construct a uniformly continuous tangent vector field, defined on the unit two-dimensional sphere and different from zero at all points belonging to the unit sphere.*

Theorem 6. *One can construct such a continuous tangent vector field F, defined on the unit two-dimensional sphere, that for every point $x\sigma y\sigma z$ belonging to the unit sphere,*

$$N(F(x\sigma y\sigma z)) = 1,$$

where N is the algorithm for computing the norm in the space $\overline{\mathcal{E}}_{3,2}$.

Using the operator H from the proof of Theorem 3, construct a constructive uniformly continuous mapping H_1 of the closed upper unit hemisphere into itself, without a fixed point, shifting all points

of the equator upward along the meridians. We transform the mapping H_1 in the natural way into a constructive tangent uniformly continuous vector field J_1 on the closed upper hemisphere. We now construct on the whole unit two-dimensional sphere such a vector field J that, whatever the point $x\sigma y\sigma z$ may be:

1) if $x\sigma y\sigma z$ belongs to the closed upper hemisphere, then

$$J(x\sigma y\sigma z) = J_1(x\sigma y\sigma z);$$

2) if $x\sigma y\sigma z$ belongs to the lower hemisphere, then

$$J(x\sigma y\sigma z) = A(j_1(x\sigma y\sigma |z|)),$$

where A is the operator transforming a vector into the vector symmetric to it with respect to the vertical line passing through the initial point of the vector.

The operator J is a tangent uniformly continuous vector field defined on the unit two-dimensional sphere. At every point of the two-dimensional sphere it is different from zero. To prove Theorem 6, a constructive operator G_1 is constructed such that for any point $x\sigma y\sigma z$

$$G(x\sigma y\sigma z) \simeq \frac{1}{N(J(x\sigma y\sigma z))}\, J(x\sigma y\sigma z).$$

Remark 1. All constructive operators whose potential realizability is discussed in Theorems 1–6 can be made infinitely differentiable. The constructive operators G, F_0, and G_1, constructed in the proofs of Theorems 1, 2, and 6, are locally uniformly continuous operators.

Remark 2. Any retraction of the square $-1\tau 1$ onto its boundary and any normalized tangent vector field defined on the unit two-dimensional sphere do not satisfy the condition of pseudouniform continuity.

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Note: Figure translations are in progress. See original paper for figures.