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Abstract

Full Text

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ON THE ASYMPTOTIC SOLUTION OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER

(Presented by Academician N. N. Bogolyubov on 10 I 1963)

MATHEMATICS

1. Numerous works have been devoted to the construction of the asymptotic solution of systems of differential equations containing a parameter ($^{1-7}$). Fundamental results were obtained by N. N. Bogolyubov (^{1}).

Consider, in n-dimensional space, the system of differential equations*

$$\frac{dx}{dt} = A(\tau,\varepsilon)x + \varepsilon B(\tau,\varepsilon)e^{i\theta}, \tag{1}$$

where $A(\tau,\varepsilon)$ is a real matrix of order n; $B(\tau,\varepsilon)$ is an n-dimensional vector, admitting formal expansions

$$A(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A^{(s)}(\tau), \qquad B(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s B^{(s)}(\tau), \qquad 0 \leqslant \tau = \varepsilon t \leqslant L; \tag{2}$$

ε is a small real parameter.

The asymptotic solution of system (1) is determined by the behavior of the roots of the equation

$$\det\left\|A^{(0)}(\tau) - \lambda E\right\| = 0, \tag{3}$$

where E is the identity matrix.

In (^{2}) a solution was obtained for the case of simple roots on the segment $[0, L]$ of equation (3). The case of multiple roots is partially considered in (^{3,4,7}). The asymptotic splitting of system (1) is given in (^{5,6}) for arbitrary behavior of the roots of equation (3).

In the present article we consider the case when several multiple elementary divisors correspond to a multiple root.

2. Denote by $\lambda_1(\tau), \lambda_2(\tau), \dots, \lambda_p(\tau)$ the roots of equation (3), having respectively the constant multiplicities $k_1, k_2, \dots, k_p$ $(k_1 + k_2 + \dots + k_p = n)$, and suppose that to the root $\lambda_j(\tau)$ $(j = 1, \dots, p)$ there correspond r_j elementary divisors
$$(\lambda - \lambda_j(\tau))^{s_{j1}}, \dots, (\lambda - \lambda_j(\tau))^{s_{jr_j}}$$
$(s_{j1} + \dots + s_{jr_j} = k_j;\ j = 1, \dots, p)$. Then one can indicate a nonsingular matrix $T(\tau)$ such that
$$T^{-1}(\tau)A^{(0)}(\tau)T(\tau) = W(\tau) = \begin{Vmatrix} W_{k_1}(\tau) & & 0 \\ & \ddots & \\ 0 & & W_{k_p}(\tau) \end{Vmatrix}, \tag{4}$$

where
$$W_{k_j}(\tau) = \begin{Vmatrix} W_{s_{j1}}(\tau) & & 0 \\ 0 & \ddots & \\ & & W_{s_{jr_j}}(\tau) \end{Vmatrix}, \quad W_{s_{jn_j}}(\tau) = \left.\begin{Vmatrix} \lambda_j(\tau) & 1 & 0 & \dots & 0 \\ 0 & \lambda_j(\tau) & 1 & 0 \dots 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \lambda_j(\tau) & \end{Vmatrix}\right\} s_{jn_j}, \tag{5}$$
$$n_j = 1, \dots, r_j; \quad j = 1, \dots, p;$$

$T^{-1}(\tau)$ is the matrix inverse to the matrix $T(\tau)$.

Suppose that the function $i\omega(\tau)$ $(\omega(\tau) = d\theta/dt,\ i = \sqrt{-1})$ at isolated points of the segment $[0, L]$ becomes equal to one of the roots of equation (3), for example $\lambda_1(\tau)$, but $i\omega(\tau) \neq \lambda_k(\tau)$ $(k = 2, \dots, p)$ for any $\tau \in [0, L]$. This case is customarily called "resonant."

Theorem 1. *If $A(\tau, \varepsilon)$, $B(\tau, \varepsilon)$, $\omega(\tau)$ have derivatives of all orders with respect to τ, and for all $\tau \in [0, L]$ the ν_{jn_j}-component of the vector*
$$C(\tau) = T^{-1}(\tau)\left[A^{(1)}(\tau)T_{\nu_{jn_j-1}+1}(\tau) - \frac{dT_{\nu_{jn_j-1}+1}(\tau)}{d\tau}\right], \tag{6}$$
where $T_{\nu_{jn_j-1}+1}(\tau)$ is the column of the matrix $T(\tau)$ numbered $\nu_{jn_j-1}+1$ $(\nu_{jn_j} = k_1 + \dots + k_{j-1} + s_{j1} + \dots + s_{jn_j};\ n_j = 1, \dots, r_j;\ j = 1, \dots, p)$, is not equal to zero, then the asymptotic solution of system (1) in the "resonant" case can be represented in the form

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* Many differential equations are reducible to a system of the form (1), in particular differential equations with a small parameter at the highest derivatives.

$$x = \sum_{n_1=1}^{r_1} \left[U_{1n_1}(\tau, \mu_{1n_1}) h_{1n_1} + P_{n_1}(\tau, \mu_{1n_1}) \right] e^{i\theta} +$$

$$+ \sum_{k=2}^{p} \sum_{n_k=1}^{r_k} U_{kn_k}(\tau, \mu_{kn_k}) h_{kn_k}; \tag{7}$$

$$\frac{dh_{1n_1}}{dt} = [\lambda_{1n_1}(\tau, \mu_{1n_1}) - i\omega(\tau)] h_{1n_1} + z_{n_1}(\tau, \mu_{1n_1}), \quad n_1 = 1, \dots, r_1; \tag{8}$$

$$\frac{dh_{kn_k}}{dt} = \lambda_{kn_k}(\tau, \mu_{kn_k}) h_{kn_k}, \quad n_k = 1, \dots, r_k; \quad k = 2, \dots, p, \tag{9}$$

where $U_{jn_j}(\tau, \mu_{jn_j})$, $P_{n_1}(\tau, \mu_{1n_1})$ *are* n*-dimensional vectors;* h_{jn_j}, $z_{n_1}(\tau, \mu_{1n_1})$ *are scalar functions* $(n_j = 1, \dots, r_j;\ j = 1, \dots, p)$, *admitting formal expansions*

$$U_{jn_j}(\tau, \mu_{jn_j}) = \sum_{s=0}^{\infty} \mu_{jn_j}^{s} U_{jn_j}^{(s)}(\tau), \quad \lambda_{jn_j}(\tau, \mu_{jn_j}) = \lambda_j(\tau) + \sum_{s=1}^{\infty} \mu_{jn_j}^{s} \lambda_{jn_j}^{(s)}(\tau); \tag{10}$$

$$P_{n_1}(\tau, \mu_{1n_1}) = \sum_{s=0}^{\infty} \mu_{1n_1}^{s} P_{n_1}^{(s)}(\tau), \quad z_{n_1}(\tau, \mu_{1n_1}) = \sum_{s=0}^{\infty} \mu_{1n_1}^{s} z_{n_1}^{(s)}(\tau); \tag{11}$$

$$\mu_{jn_j} = \sqrt[s_{jn_j}]{\varepsilon}, \quad n_j = 1, \dots, r_j; \quad j = 1, \dots, p. \tag{12}$$

Substituting the vector x, determined by relations (7)–(9), into system (1) and, in the identity obtained, equating the coefficients of the functions h_{jn_j} and the free terms, we have

$$(A - \lambda_{jn_j} E) U_{jn_j} = \varepsilon U'_{jn_j}, \quad n_j = 1, \dots, r_j; \quad j = 1, \dots, p^{*}, \tag{13}$$

$$(A - i\omega E) P_{n_1} = U_{1n_1} z_{n_1} + \varepsilon (P'_{n_1} - B), \quad n_1 = 1, \dots, r_1. \tag{14}$$

* Here and below we omit the arguments of the quantities.

To determine the coefficients of the series (10), we shall use relation (13). Equating in it the coefficient of $\mu_{jn_j}^{0}$, we have

$$(A^{(0)} - \lambda_j E) U_{in_j}^{(0)} = 0, \qquad n_j = 1, \dots, r_j;\ j = 1, \dots, p. \tag{15}$$

Introducing the vector

$$Q^{(s)}_{jn_j} = T^{-1} U^{(s)}_{jn_j}, \qquad s = 0, 1, \dots, \tag{16}$$

equation (15) may be put in the form

$$(W - \lambda_j E) Q^{(0)}_{jn_j} = 0, \tag{17}$$

or, according to (4),

$$(W_{k_r} - \lambda_j E) Q^{(0)}_{jn_j k_r} = 0, \qquad n_j = 1, \dots, r_j;\ r, j = 1, \dots, p, \tag{18}$$

where $Q^{(0)}_{jn_j k_r}$ is the vector with components

$$Q^{(0)}_{jn_j k_r} = (\{q^{(0)}_{jn_j}\}_{l_{r-1}+1},\ \{q^{(0)}_{jn_j}\}_{l_{r-1}+2}, \dots, \{q^{(0)}_{jn_j}\}_{l_r}), \tag{19}$$

$$l_r = k_1 + \dots + k_r; \quad r = 1, \dots, p.$$

From (18) we find

$$Q^{(0)}_{jn_j k_r} \equiv 0, \qquad j \neq r;\ j, r = 1, \dots, p;\ n_j = 1, \dots, r_j. \tag{20}$$

For $r = j$, equation (18) is representable in the form

$$(W_{s_j i_j} - \lambda_j E) Q^{(0)}_{jn_j s_j i_j} = 0, \tag{21}$$

where

$$Q^{(0)}_{jn_j s_j i_j} = (\{q^{(0)}_{jn_j}\}_{v_{i_j j-1}+1},\ \{q^{(0)}_{jn_j}\}_{v_{i_j j-1}+2}, \dots, \{q^{(0)}_{jn_j}\}_{v_{i_j j}}), \tag{22}$$

$$v_{i_j j} = k_1 + \dots + k_{j-1} + s_{j1} + \dots + s_{j i_j};$$

$$n_j, i_j = 1, \dots, r_j; \quad j = 1, \dots, p.$$

Taking (5) into account, from (21) we find

$$\{q^{(0)}_{jn_j}\}_{v_{i_j j-1}+2} = \dots = \{q^{(0)}_{jn_j}\}_{v_{i_j j}} = 0, \qquad n_j, i_j = 1, \dots, r_j;\ j = 1, \dots, p. \tag{23}$$

The component $\{q^{(0)}_{jn_j}\}_{v_{i_j j-1}+1}$ is arbitrary. Put

$$\{q^{(0)}_{jn_j}\}_{v_{i_j j-1}+1} = \begin{cases} 1, & i_j = n_j, \\ 0, & i_j \neq n_j;\ i_j, n_j = 1, \dots, r_j;\ j = 1, \dots, p. \end{cases} \tag{24}$$

To determine the vector $Q^{(1)}_{jn_j}$ and the function $\lambda^{(1)}_{jn_j}$, compare in (13) the coefficients of $\mu^{1}_{jn_j}$. Then, repeating the preceding arguments, we find:

$$Q^{(1)}_{jn_j k_r} \equiv 0, \qquad Q^{(1)}_{jn_j s_j i_j} \equiv 0, \qquad r \neq j;\ i_j \neq n_j;\ i_j, n_j = 1, \dots, r_j; \tag{25}$$

$$r, j = 1, \dots, p;$$

$$\{q^{(1)}_{jn_j}\}_{v_{jn_j-1}+2} = \lambda^{(1)}_{jn_j}, \qquad \{q^{(1)}_{jn_j}\}_{v_{jn_j-1}+3} = \dots = \{q^{(1)}_{jn_j}\}_{v_{jn_j}} = 0. \tag{26}$$

The component $\{q^{(1)}_{jn_j}\}_{v_{jn_j-1}+1}$ is arbitrary. We put it equal to zero. Equating now in relation (13) the coefficients successively of $\mu^{2}_{jn_j}, \dots, \mu^{s_{jn_j}-1}_{jn_j}$ and applying the method of induction, we have

$$\{q^{(s_{jn_j}-1)}_{jn_j}\}_{v_{jn_j}} = (\lambda^{(1)}_{jn_j})^{s_{jn_j}-1}, \qquad \{q^{(\eta_{jn_j})}_{jn_j}\}_{v_{jn_j}} = 0, \qquad 0 \leq \eta_{jn_j} \leq s_{jn_j} - 2. \tag{27}$$

Then, equating in (13) the coefficients of $\mu^{s_{jn_j}}_{jn_j}$ and taking (27) into account, we find

$$\lambda^{(1)}_{in_j}(\tau) = \sqrt[s_{jn_j}]{\left\{ T^{-1}(\tau) \left[A^{(1)}(\tau) T_{\nu_{jn_j}-1+1}(\tau) - \frac{dT_{\nu_{jn_j}-1+1}(\tau)}{dt} \right] \right\}_{\nu_{jn_j}}}, \tag{28}$$

$$n_j = 1, \dots, r_j; \quad j = 1, \dots, p.$$

All subsequent coefficients of the series (10) are determined in the same way.

The vector $P_{n_1}(\tau, \mu_{1n_1})$ and the function $z_{n_1}(\tau, \mu_{1n_1})$ $(n_1 = 1, \dots, r_1)$ are determined from relation (14).

Applying the method set forth in (^{1}), one can prove the following theorem.

Theorem 2. *If the conditions of Theorem 1 are satisfied and*

$$x\big|_{t=0} = x_m\big|_{t=0}; \qquad \operatorname{Re}\left(\sum_{s=0}^{s_{jn_j}-1} \mu_{jn_j}^{s} \lambda_{jn_j}^{(s)}\right) \le 0, \quad n_j = 1, \dots, r_j; \quad j = 1, \dots, p, \tag{29}$$

where x_m is the vector defined by relations (7)–(12); if the series are cut off after the m-th terms, then for any $L > 0$ and $0 < \mu_{jn_j} \le \bar{\mu}_{jn_j}$ one can indicate constants C_{jn_j}, independent of the parameter μ_{jn_j}, such that

$$\|x - x_m\| \le \sum_{j=1}^{p} \sum_{n_j=1}^{r_j} \mu_{jn_j}^{m+2-2s_{jn_j}} C_{jn_j}. \tag{30}$$

Relation (30) proves the asymptotic character of the solution x_m.

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Note: Figure translations are in progress. See original paper for figures.