On polynomials of best (quadratic) approximation in a prescribed system of points

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. D. Kalashnikov, G. P. Gubanov

On polynomials of best (quadratic) approximation in a prescribed system of points

(Presented by Academician P. S. Novikov on March 8, 1963)

1. Let $\bar{C}$ be the space of all continuous 2π-periodic, with respect to each of the variables x and y, functions $f(x, y)$, and let $S_{mn}(f; x, y)$ be the Fourier sum of order (m, n) of the function $f(x, y) \in \bar{C}$. It is well known that, among all trigonometric polynomials $T_{mn}(x, y)$ of order not exceeding (m, n),

$$
\begin{aligned}
T_{mn}(x, y) = \frac{\alpha_{00}}{4} &+ \frac{1}{2}\sum_{\mu=1}^{m}(\alpha_{\mu 0}\cos\mu x + \gamma_{\mu 0}\sin\mu x) \\
&+ \frac{1}{2}\sum_{\nu=1}^{n}(\alpha_{0\nu}\cos\nu y + \beta_{0\nu}\sin\nu y) + \sum_{\mu=1}^{m}\sum_{\nu=1}^{n}(\alpha_{\mu\nu}\cos\mu x\cos\nu y \\
&+ \beta_{\mu\nu}\cos\mu x\sin\nu y + \gamma_{\mu\nu}\sin\mu x\cos\nu y + \delta_{\mu\nu}\sin\mu x\sin\nu y)
\end{aligned}
$$

the least value of the integral

$$
\int_0^{2\pi}\int_0^{2\pi}|f(x, y) - T_{mn}(x, y)|\,dx\,dy
$$

is delivered by $S_{mn}(f; x, y)$.

Let a system of equidistant points (x_i, y_j) be given, where $x_i = 2i\pi/M$, $i = 1, 2, \dots, M$; $y_j = 2j\pi/N$, $j = 1, 2, \dots, N$; M and N are arbitrary natural numbers greater than $2m + 1$ and $2n + 1$, respectively.

It is easy to verify that, among all trigonometric polynomials $T_{mn}(x, y)$ of order not exceeding (m, n), the least value of the sum

$$
\sum_{i=1}^{M}\sum_{j=1}^{N}[f(x_i, y_j) - T_{mn}(x_i, y_j)]^2
$$

is delivered by the polynomial with coefficients

$$\alpha_{\mu\nu} = \frac{4}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} f(x_i, y_j) \cos \mu x_i \cos \nu y_j,$$

$$\beta_{\mu\nu} = \frac{4}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} f(x_i, y_j) \cos \mu x_i \sin \nu y_j,$$

$$\gamma_{\mu\nu} = \frac{4}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} f(x_i, y_j) \sin \mu x_i \cos \nu y_j,$$

$$\delta_{\mu\nu} = \frac{4}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} f(x_i, y_j) \sin \mu x_i \sin \nu y_j$$

$$(\mu = 0, 1, 2, \dots, m;\ \nu = 0, 1, 2, \dots, n).$$

We shall call this polynomial the polynomial of best (quadratic) approximation for the function $f(x, y)$ in the system of points (x_i, y_j), where $x_i = 2\pi i/M$, $i = 1, 2, \dots, M$; $y_j = 2\pi j/N$, $j = 1, 2, \dots, N$, and shall denote it by

$$T_{mn}^{MN}(f; x, y).$$

Substituting the values of the coefficients $\alpha_{\mu\nu}, \beta_{\mu\nu}, \gamma_{\mu\nu}, \delta_{\mu\nu}$, we obtain for the polynomial $T_{mn}^{MN}(f; x, y)$ the expression

$$T_{mn}^{MN}(f; x, y) = \frac{4}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} f(x_i, y_j) D_m(x_i - x) D_n(y_j - y),$$

where

$$D(t) = \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2} t}$$

is the Dirichlet kernel.

If $M = 2m + 1$, $N = 2n + 1$, then $T_{mn}^{MN}(f; x, y)$ turns into the interpolating trigonometric polynomial $\widetilde{S}_{mn}(f; x, y)$ for the function $f(x, y)$ with equidistant interpolation nodes. If $M = r(2m + 1)$, $N = s(2n + 1)$, then in the limit as $r, s \to \infty$ we obtain the Fourier sum $S_{mn}(f; x, y)$.

2. Let H_{ω_1, ω_2} denote the class of functions $f(x, y)$ of period 2π with respect to x and y, satisfying the condition

$$|f(x_1, y_1) - f(x_2, y_2)| \le \omega_1(|x_1 - x_2|) + \omega_2(|y_1 - y_2|),$$

where $\omega_1(t)$ and $\omega_2(z)$ are convex moduli of continuity. By $\mathcal{E}_{mn}^{(r,s)}(x, y)$ we denote the exact least upper bound of the deviations of the functions $f(x, y)$ from the polynomials $T_{mn}^{r(2m+1)s(2n+1)}(f; x, y)$, extended over the entire class H_{ω_1,ω_2}:

$$\mathcal{E}_{mn}^{(r,s)}(x, y) = \sup_{f \in H_{\omega_1,\omega_2}} |f(x, y) - T_{mn}^{r(2m+1)s(2n+1)}(f; x, y)|.$$

It is not difficult to verify that the upper bound $\mathcal{E}_{mn}^{(r,s)}(x, y)$ is an even function, periodic with period $h^{(r)} = 2\pi/r(2m + 1)$ with respect to x and with period $g^{(s)} = 2\pi/s(2n + 1)$ with respect to y.

Theorem. For any integers $r, s > 1$ the following asymptotic equality holds

$$\mathcal{E}_{mn}^{(r,s)}(x, y) = \frac{8 \ln m \ln n}{\pi^2 rs} \left| \cos \frac{2m + 1}{2} x \cos \frac{2n + 1}{2} y \right| \times \sum_{i=1}^{[r/2]} \sum_{j=1}^{[s/2]} \min\{\omega_1(2x_i),\ \omega_2(2y_j)\} \sin \frac{i\pi}{r} \sin \frac{j\pi}{s} + \rho_{mn}$$

$$\left(0 \le x \le \tfrac{1}{2} h^{(r)}, \quad 0 \le y \le \tfrac{1}{2} g^{(s)}\right),$$

where

$$\rho_{mn} = O\left\{(\ln m + \ln n) \left[\omega_1\left(\frac{2\pi}{2m + 1}\right) + \omega_2\left(\frac{2\pi}{2n + 1}\right)\right]\right\}, \qquad x_i = ih^{(r)}, \quad y_j = jg^{(s)}.$$

Passing to the limit as $r, s \to \infty$, from the theorem we obtain the result of P. T. Bugaits ([1]):

$$\sup_{f \in H_{\omega_1,\omega_2}} |f(x, y) - S_{mn}(f; x, y)| = \frac{2(2m + 1)(2n + 1) \ln m \ln n}{\pi^4} \times \int_0^{\pi/(2m+1)} \int_0^{\pi/(2n+1)} \min\{\omega_1(2u), \omega_2(2v)\} \sin \frac{2m + 1}{2} u \sin \frac{2n + 1}{2} v \, du \, dv +$$

where

$$\rho_{mn} = O\left\{(\ln m + \ln n) \left[\omega_1\left(\frac{2\pi}{2m + 1}\right) + \omega_2\left(\frac{2\pi}{2n + 1}\right)\right]\right\}.$$

3. Let us also note the following. We shall regard $T_{mn}^{MN}(f; x, y)$ as a linear operator (for fixed x, y, a linear functional) in the space $\overline{C}$, where the norm of $f(x, y)$ is taken to be $\sup_{x,y}|f(x, y)|$. Denote by L_{mn}^{MN} the norm of this operator. We have

$$L_{mn}^{MN} = \sup_{x,y} L_{mn}^{MN}(x, y), \qquad L_{mn}^{MN}(x, y) = \frac{4}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}|D_m(x_i - x)D_n(y_i - y)|.$$

Here $L_{mn}^{MN}(x, y)$ is a function periodic with period $h = 2\pi/M$ with respect to x and with period $g = 2\pi/N$ with respect to y, and is even. We indicate an expression for the norm of the operator $L_{mn}^{MN}(x, y)$.

Let

$$M = 2m + 1 + 1 + p, \quad 0 < p \leq 2m + 1; \qquad N = 2n + 1 + q, \quad 0 < q \leq 2n + 1.$$

In this case

$$\begin{aligned} L_{mn}^{MN}(x, y) = &\left(\frac{2}{\pi}\sin\frac{2m+1}{2}x\ln\frac{M}{p} + \frac{p}{\pi M\sin p\pi/2M}\sum_{\mu=1}^{p}\frac{(-1)^\mu}{\mu}\right. \\ &\times\left.\left\{\cos\left[\left(k'_\mu + \frac{1}{2}\right)\frac{\pi p}{M} + \left(m + \frac{1}{2}\right)x\right] + \cos\left[\left(k_\mu + \frac{1}{2}\right)\frac{\pi p}{M} - \left(m + \frac{1}{2}\right)x\right]\right\}\right) \\ &\times\left(\frac{2}{\pi}\sin\frac{2n+1}{2}y\ln\frac{N}{q} + \frac{q}{\pi N\sin q\pi/2N}\sum_{\nu=1}^{q}\frac{(-1)^\nu}{\nu}\left\{\cos\left[\left(l'_\nu + \frac{1}{2}\right)\frac{\pi q}{N} + \left(n + \frac{1}{2}\right)y\right]\right.\right. \\ &\left.\left.+\cos\left[\left(l_\nu + \frac{1}{2}\right)\frac{\pi q}{N} - \left(n + \frac{1}{2}\right)y\right]\right\}\right) + O\left(\ln\frac{M}{p}\right) + O\left(\ln\frac{N}{q}\right) \end{aligned}$$

$$(0 \leq x \leq \tfrac{1}{2}h,\ 0 \leq y \leq \tfrac{1}{2}g),$$

where

$$k'_\mu = \left[\mu\frac{M}{p} - \frac{2m+1}{p}\frac{x}{h}\right], \qquad k_\mu = \left[\mu\frac{M}{p} + \frac{2m+1}{p}\frac{x}{h}\right],$$

$$\mu = 1, 2, \ldots, p, \qquad h = \frac{2\pi}{M},$$

$$l'_\nu = \left[\nu\frac{N}{q} - \frac{2n+1}{q}\frac{y}{g}\right], \qquad l_\nu = \left[\nu\frac{N}{q} + \frac{2n+1}{q}\frac{y}{g}\right],$$

$$\nu = 1, 2, \dots, q, \qquad g = \frac{2\pi}{N}.$$

If $M \geq 2(2m+1)$, $N \geq 2(2n+1)$, then

$$\begin{aligned} L_{mn}^{MN}(x,y) = & \frac{uv}{\pi^2 MN \sin(\pi v/2M) \cdot \sin(\pi u/2N)} \sum_{\mu=1}^{v} \sum_{\nu=1}^{u} \frac{(-1)^{\mu+\nu}}{\mu\nu} \\ & \times \left\{ \cos\left[\left(l_\mu + \frac{1}{2} \right) \frac{\pi v}{M} - \left(m + \frac{1}{2} \right) x \right] + \cos\left[\left(l'_\mu + \frac{1}{2} \right) \frac{\pi v}{M} + \left(m + \frac{1}{2} \right) x \right] \right\} \\ & \times \left\{ \cos\left[\left(k_\nu + \frac{1}{2} \right) \frac{\pi u}{N} - \left(n + \frac{1}{2} \right) y \right] + \cos\left[\left(k'_\nu + \frac{1}{2} \right) \frac{\pi u}{N} + \left(n + \frac{1}{2} \right) y \right] \right\} \\ & + O(\ln m) + O(\ln n) \end{aligned}$$

$$(0 \leq x \leq \tfrac{1}{2}h, \qquad 0 \leq y \leq \tfrac{1}{2}g),$$

where

$$l_\mu = \left[\mu \frac{M}{v} + \frac{x}{h} \right], \quad l'_\mu = \left[\mu \frac{M}{v} - \frac{x}{h} \right], \quad \mu = 1, 2, \dots, v, \quad v = 2m+1,$$

$$h = \frac{2\pi}{M}; \qquad k_\nu = \left[\nu \frac{N}{u} + \frac{y}{g} \right], \quad k'_\nu = \left[\nu \frac{N}{u} - \frac{y}{g} \right], \quad \nu = 1, 2, \dots, u,$$

$$u = 2n+1, \qquad g = \frac{2\pi}{N}.$$

The corresponding equalities can be written for the cases when $2m+1 < M \leq 2(2m+1)$, $N \geq 2(2n+1)$ and $2n+1 < N \leq 2(2n+1)$, $M \geq 2(2m+1)$.

If $M = r(2m+1)$, $N = s(2n+1)$, where r, s are integers, then

$$L_{mn}^{r(2m+1)s(2n+1)}(x,y) = \frac{4 \ln m \ln n}{\pi^2} \frac{\cos\left[\frac{\pi}{2r} - \frac{2m+1}{2}x\right] \cos\left[\frac{\pi}{2s} - \frac{2n+1}{2}y\right]}{rs \sin(\pi/2r) \cdot \sin(\pi/2s)} + O(\ln m) + O(\ln n),$$

whence, putting $x = 0$, $y = 0$ and passing to the limit as $r, s \to \infty$, we obtain the expression for the Lebesgue constant

$$L_{mn} = \frac{16}{\pi^4} \ln m \ln n + O(\ln m) + O(\ln n).$$

Dnepropetrovsk State University
named after the 300th anniversary of the reunification of Ukraine with Russia

Received
26 IV 1962

CITED LITERATURE

1. P. T. Bugaets, DAN, **79**, 557 (1951).

Note: Figure translations are in progress. See original paper for figures.