



Soviet-era science, translated into English

O. N. GOLOVIN

1963

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.57056>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

O. N. GOLOVIN

FUNCTORIAL OPERATIONS ON THE CLASS OF ALL GROUPS

(Presented by Academician P. S. Aleksandrov, 11 X 1962)

In group theory in recent years ever new operations on groups have been described, and various products of groups have been constructed. From all this multitude of operations I single out **exact** operations.

Definition ⁽⁴⁾. We shall say that an **exact operation** \circ is given on the class of all groups if to every set of groups $G_\alpha, \alpha \in I$ (among which there may also be isomorphic ones) there is assigned uniquely a group G , called their \circ -product and denoted by

$$\prod_{\alpha \in I}^{\circ} G_\alpha,$$

where: a) $G = \{G_\alpha^* \mid \alpha \in I\}$, where $G_\alpha^* \cong G_\alpha$; b) concrete isomorphisms $G_\alpha \cong G_\alpha^*, \alpha \in I$, are fixed, in the sense of which the factors G_α are regarded as embedded in G ; c) for any set of groups $K_\alpha, K_\alpha \cong G_\alpha, \alpha \in I$, there exists such a set of isomorphisms $\psi_\alpha : G_\alpha \rightarrow K_\alpha, \alpha \in I$, which are glued together into an isomorphism

$$\psi : G \rightarrow K = \prod_{\alpha \in I}^{\circ} K_\alpha.$$

It is precisely in this way, including concrete embeddings of the factors in the products, that direct and free products, which are the simplest examples of exact operations, are understood in group theory. It follows from the definition that exact operations are commutative: the set I of indices is not assumed to be ordered. In the process of studying exact operations and constructing concrete types of such operations, various authors established certain general properties of these operations, or else introduced in advance restrictions of a certain type imposed on them. I collect the main ones into the following groups of postulates, including also some postulates formulated (as far as I know) for the first time.

I group–internal postulates

They are imposed on a separate product

$$G = \prod_{\alpha \in I}^{\circ} G_\alpha.$$

I-1. Regularity postulate ⁽¹⁾: for every $\alpha \in I$

$$G_\alpha \cap \overline{D}_\alpha = E,$$

where $D_\alpha = \{G_\beta \mid \beta \in I \setminus \alpha\}$, and the bar over the letter denoting a subgroup means passage to the normal divisor generated by it.

I-2. Strengthened regularity postulate ⁽²⁾: for

$$I = I_1 \cup I_2, \quad I_1 \cap I_2 = \emptyset, \quad \{G_\alpha \mid \alpha \in I_1\} \cap \{\overline{G}_\beta \mid \beta \in I_2\} = E.$$

I-3. Postulate of gluing automorphisms ⁽³⁾: every system of automorphisms

$$\varphi_\alpha : G_\alpha \rightarrow G_\alpha, \quad \alpha \in I,$$

is glued together into an automorphism

$$\varphi : G \rightarrow G.$$

I-4. Postulate of gluing endomorphisms ⁽⁵⁾: every system of endomorphisms

$$\varphi_\alpha : G_\alpha \rightarrow G_\alpha, \quad \alpha \in I,$$

is glued together into an endomorphism

$$\varphi : G \rightarrow G.$$

I-4*. Postulate of gluing endomorphisms with a kernel condition.

This postulate is a strengthening of the preceding one and is obtained from it by adding the following **kernel condition**, formulated below for a more general case.

Kernel condition. If a certain system of homomorphisms

$$\psi_\alpha : G_\alpha \rightarrow H_\alpha, \quad \alpha \in I,$$

is glued together into a homomorphism

$$\psi : G \rightarrow H = \{H_\alpha \mid \alpha \in I\},$$

then the kernel of this resulting homomorphism ψ is minimal:

$$\text{Ker } \psi =$$

$= \overline{\{\text{Ker } \psi_\alpha \mid \alpha \in I\}}$. This condition has as its content the maximally possible freedom for gluing the homomorphisms under consideration. Everywhere in what follows, the postulates with an asterisk are obtained from the corresponding postulates without asterisks by the additional imposition of the kernel condition.

II group–external postulates

They connect two products with each other: $G = \prod_{\alpha \in I}^0 G_\alpha$ and $H = \prod_{\alpha \in I}^0 H_\alpha$.

II-1. Symmetry postulate ⁽⁴⁾: any system of isomorphisms $\psi_\alpha : G_\alpha \rightarrow H_\alpha$, $\alpha \in I$, is glued into an isomorphism $\psi : G \rightarrow H$.

II-2. Postulate of gluing epimorphisms ⁽⁶⁾. It is obtained from II-1 if arbitrary epimorphisms are understood under ψ_α and ψ .

II-2* . Postulate of gluing epimorphisms with the kernel condition (Mac Lane' s postulate) ⁽⁵⁾.

II-2* . Original version of Mac Lane' s postulate ⁽⁷⁾: if N_α are normal divisors in G_α and $N = \overline{\{N_\alpha \mid \alpha \in I\}}$, then there exists a natural isomorphism $G/N \cong G^* = \prod_{\alpha \in I}^0 (G_\alpha/N_\alpha)$.

II-3. Postulate of gluing monomorphisms (weakened Mal' cev postulate) ^(1,5). It is obtained from II-1 if arbitrary monomorphisms are understood under ψ_α and ψ .

II-3* . Postulate of gluing monomorphisms with the kernel condition (Mal' cev' s postulate) ^(8,3).

II-4. Functoriality postulate. It is obtained from II-1 if arbitrary homomorphisms are understood under ψ_α and ψ .

After fixing the number of factors, such **functorial operations** turn into functors on the category of groups, covariant in all arguments.

II-4* . Postulate of functoriality with the kernel condition.

II-4* . Postulate of functoriality with the kernel condition imposed only on epimorphisms.

II-4* . Postulate of functoriality with the kernel condition imposed only on monomorphisms.

II-5. Postulate of eliminability of identity factors: if among G_α , $\alpha \in I$, the groups G_β , $\beta \in J \subset I$, are identity groups, then the system of identity automorphisms $\varphi_\alpha : G_\alpha \rightarrow G_\alpha$, $\alpha \in I \setminus J$, is glued into an isomorphism

$$\varphi : \prod_{\alpha \in I}^0 G_\alpha \rightarrow \prod_{\gamma \in I \setminus J}^0 G_\gamma.$$

In other words, the product does not change when identity factors are removed from it.

As far as I know, the last five postulates have not been formulated in the literature (at least, not explicitly).

III group–postulates of associativity

III-1. Localization postulate: if $G = \prod_{\alpha \in I}^0 G_\alpha$ and $J \subset I$, then

$$\{G_\alpha \mid \alpha \in J\}_G = \prod_{\alpha \in J}^0 G_\alpha.$$

III-2. Blocking postulate: if $G = \prod_{\alpha \in I}^0 G_\alpha$ and

$$H_\beta = \{G_\alpha \mid \alpha \in I_\beta\}_G,$$

where

$$I = \bigcup_{\beta \in J} I_\beta, \quad I_\beta \cap I_{\beta'} = \emptyset \quad \text{for } \beta \neq \beta',$$

then

$$G = \prod_{\beta \in J}^0 H_\beta.$$

III-3. Transitivity postulate: if $G = \prod_{\alpha \in I}^0 G_\alpha$ and

$$G_\alpha = \prod_{\beta \in J_\alpha}^0 G_{\alpha,\beta}, \quad \alpha \in I,$$

then

$$G = \prod_{\beta \in J_\alpha, \alpha \in I}^0 G_{\alpha,\beta}.$$

By the very construction, from each postulate with an asterisk there follows the corresponding postulate without the asterisk. It is also clear that from I-2 there follows I-1, from II-2* there follows II-2*st, and from II-4* there follow II-4*e and II-4*m. There are numerous other relations among the postulates. I give some of the dependences known to me.

From the system II-4*, II-5 there follow all the postulates of groups I and II and postulate III-1. Postulate II-4* is equivalent to the system of postulates II-2* and II-3*. I know of no functor operations different from direct and free products (i.e. operations satisfying postulates II-4* and II-5).

From the system II-4*e, II-5 there follow all postulates except I-4*, II-3*, II-4*, II-4*m, III-2, and III-3. Postulate II-4*e is equivalent to the system of postulates II-2* and II-3, and the system II-4*e, II-5—to the system I-4, II-2*, II-5. Every episurjective functor operation (i.e. an operation satisfying postulates II-4*e and II-5) is a neutral VV -operation ⁽⁵⁾ and conversely. In particular, all verbal operations of Moran ^(9, 6) are episurjective functor operations.

From postulate II-2 there follows neither II-2* nor II-2*st, so that the kernel condition is independent. From the system I-1, II-1 there does not follow II-5: there exist regular symmetric operations with respect to which the identity factors do not play the role of an identity. Postulate II-2* is equivalent to the system II-1, II-2*st. The system of postulates II-1, III-3 is equivalent to the system II-1, III-1, III-2.

From what has been said it follows that the whole collection of postulates of groups I, II, and III is equivalent to the system II-4*, II-5, III-2.

We introduce two more new postulates:

IV. **General postulate:** if

$$G = \prod_{\alpha \in I}^0 G_{\alpha} \quad \text{and} \quad H = \prod_{\beta \in J}^0 H_{\beta},$$

then any system of homomorphisms

$$\psi_{\alpha} : G_{\alpha} \rightarrow \{H_{\beta} \mid \beta \in J_{\alpha}\}_H, \quad \alpha \in I,$$

where $J_{\alpha} \subset J$ and $J_{\alpha} \cap J_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$, is glued into a homomorphism $\psi : G \rightarrow H$.

IV*. **General postulate with kernel condition.**

Exact operations satisfying this postulate IV* will be called **perfect**. The direct and free products are perfect operations. We shall regard the direct product as a **trivial perfect** operation.

Theorem 1. *Postulate IV* is equivalent to the system of postulates II-4*, II-5, and III-2 and therefore is equivalent to all postulates of groups I, II, and III. In other words, perfect operations are the same as associative free functor operations.*

In postulate IV* (and the requirements included in the definition of an exact operation) are collected all the general properties known to me of direct and free products. It is unknown whether there exist perfect operations different from direct and free products. Up to now the Mal'cev problem (^{8,3,10}) on the existence of associative regular operations satisfying postulate II-3*, different from direct and free products, also remains open. The facts given below, concerning perfect operations, show, in any case, that all nontrivial perfect operations (if such, different from the free product, exist) differ essentially in their properties from the direct product and, conversely, are close in a number of qualities to the free product.

Theorem 2. Direct multiplication can be defined as a proper localizable exact operation acting on pairs of groups according to the law of direct multiplication. This property completely characterizes it among all perfect operations. Moreover, whatever nontrivial perfect operation \circ may be and whatever natural

number n , $n \geq 2$, there exists such an episfree functorial operation \odot that the actions of both operations on any n factors (and, consequently, on a smaller number of factors as well) coincide, while there will be $n + 1$ groups whose \circ - and \odot -products are already different.*

Theorem 3. A free functorial (in particular, perfect) operation whose action coincides with the action of direct multiplication on at least one pair of nonidentity groups is direct multiplication**.

Theorem 4. If \circ is a nontrivial perfect operation, then no nonidentity proper \circ -factor of any group G can contain a normal divisor of the group G different from E .

Theorem 5. Whatever natural number n may be, direct multiplication is the only perfect operation with respect to which the variety of nilpotent groups of class n is closed.

Moscow State University
named after M. V. Lomonosov

Received
9 X 1962

CITED LITERATURE

- ¹ O. N. Golovin, Matem. sborn., **27** (69), No. 3, 427 (1950).
- ² M. Benado, Math. Nachr., **14**, No. 4-6, 213 (1955-1956).
- ³ A. L. Shmelkin, Matem. sborn. **51** (93), No. 3, 277 (1960).
- ⁴ O. N. Golovin, UMN, **16**, No. 2 (98), 204 (1961).
- ⁵ O. N. Golovin, DAN, **145**, No. 5, 967 (1962).
- ⁶ S. Morgan, Proc. London Math. Soc. (3), **8**, No. 32, 548 (1958).
- ⁷ R. R. Struik, Trans. Am. Math. Soc., **81**, No. 2, 425 (1956).
- ⁸ O. N. Golovin, Matem. sborn., **28** (70), No. 2, 431 (1951).
- ⁹ S. Morgan, Proc. London Math. Soc. (3), **6**, No. 24, 581 (1956).
- ¹⁰ S. Morgan, Bull. Acad. polon. sci., Sér. sci. math., astron. et phys., **9**, No. 12, 853 (1961).

* The last assertion of this theorem was first proved by me for free multiplication, and subsequently by A. L. Shmelkin for any nontrivial perfect operation when $n = 2$.

** This theorem refines one result of A. L. Shmelkin ⁽³⁾.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.