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Abstract

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ON BOUNDARY-VALUE PROBLEMS FOR PARABOLIC SYSTEMS IN DOMAINS OF GENERAL FORM

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In the notes ([1,2]) the results of a study were set forth concerning half-space fundamental matrices of solutions (f.m.s.) of general boundary-value problems for Petrovsky parabolic systems with constant coefficients and operators homogeneous in the parabolic sense. In doing so, sharp estimates for the f.m.s. were established, the possibility of its continuation to the half-space $x_n < 0$ was proved, and the region of its possible singularities in the hyperplane $t = 0$ was indicated. This makes it possible to pass to the investigation of the solvability of boundary-value problems for systems with variable coefficients in domains of a very general form (nonconvex and noncylindrical) with smooth boundaries. Several results obtained in this direction are contained in the present paper. Here, alongside interior problems, exterior ones are also investigated. The study of general boundary-value problems for finite convex domains is the subject of the works of T. Ya. Zagorsky ([3]); his assumptions and estimates differ from ours. We use here the methods developed in the elliptic case by Ya. B. Lopatinskii ([4]), and certain constructions of S. P. Gavelya ([5]).

1. In this section we shall study boundary-value problems for systems of general form with boundary operators of equal order in nonconvex cylindrical domains. The problem under consideration is as follows: to find a solution $u(t,x)$ of the parabolic system

$$L(u) \equiv \frac{\partial u}{\partial t} - \sum_{|k|=2b} A_k(t,x)(-iD_x)^k u - \sum_{|k|\le 2b-1} A_k(t,x)(-iD_x)^k u,$$

$$\frac{\partial u}{\partial t} - A_0(t,x,-iD_x)u - A_1(t,x,-iD_x)u = 0 \qquad (1)$$

in the cylindrical (generally speaking, nonconvex) domain $G = V \times (0,T]$, satisfying the conditions:

$$u(t,x)\big|_{t=+0}=0, \quad \lim_{\substack{x\to\xi\in S\\ x\in V}} B_i(t,\xi,-iD_x)u \equiv \lim_{\substack{x\to\xi\in S\\ x\in V}} \sum_{j=1}^{N}\sum_{|k|\le r_j} B_{ij}^{(k)}(t,\xi,-iD_x)u_j \equiv$$

$$\equiv \lim_{\substack{x\to\xi\in S\\ x\in V}} \left(\sum_{j=1}^{N}\sum_{|k|=r_j} B_{ij}^{(k)}(t,\xi,-iD_x)u_j + \sum_{j=1}^{N}\sum_{|k|<r_j} B_{ij}^{(k)}(t,\xi,-iD_x)u_j\right) =$$

$$= \lim_{\substack{x\to\xi\in S\\ x\in V}} (B_{0i}u + B_{1i}u) = f_i(t,\xi), \tag{2}$$

where S is the boundary of the domain V. We shall assume that, for problem (1)–(2), condition (15) of ([2]) is fulfilled. In that work, for the problem

$$\frac{\partial u}{\partial t} = A_0(\tau,\xi,-iD_x)u, \tag{3}$$

$$u\big|_{t=+0}=0, \qquad \lim_{\substack{x\to z\\ (z,\nu(\xi))=0}} B_0(t,\xi,-iD_x)u = f(t,z), \tag{4}$$

where $\nu(\xi)$ is the inward normal at the point ξ of the boundary S of the domain V, in the half-space $(x,\nu(\xi))>0$, the existence of an f.m.s. $G(t,\tau,x,\xi)$ was proved with the following estimates for its j-th column:

$$\left|\frac{\partial^{m_0}}{\partial t^{m_0}} D_x^m G_j(t,\tau,x,\xi)\right| \le C_{m_0,m} t^{-\frac{n-1+2b-r_j+2bm_0+|m|}{2b}} \exp\left\{-C\left|\frac{x}{t^{1/2b}}\right|^{\frac{2b}{2b-1}}\right\}. \tag{5}$$

With the aid of this f.m.s., under the assumption that the domain V is convex, a special matrix of solutions (s.m.s.) $E(t,\tau,x,\xi)$ was constructed, by means of which the boundary-value problem is reduced to the solution of a Volterra integral equation of the second kind with a quasiregular kernel. The columns of this matrix are determined as follows:

$$E_j(t,\tau,x,\xi) = G_j^*(t-\tau,x-\xi,\tau,\xi)-$$
$$-\int_\tau^t d\beta \int_V Z(t,\beta,x,y)L[G_j^*(\beta-\tau,y-\xi,\tau,\xi)]\,dy, \tag{6}$$

where

$$G_j^* = \sum_{l=r_j+1}^{2b} G_j^{(l)}(t,\tau,x,\xi), \qquad G_j^{(l)}(t,\tau,x,\xi) = \int e^{i(x',\sigma')}d\sigma' \int_\Gamma e^{pt}dp \int e^{ix_n\sigma_n}\times$$

$$\times Q_j^{(l)}(p,\sigma)\,d\sigma_n;$$

$Q_j^{(l)}(p,\sigma)$ are generalized homogeneous functions of p and σ of degree $-r_j - l$ with the same analytic properties as $Q_j(p,\sigma)$ (1); $Z(t,\beta,x,y)$ is the fundamental matrix of solutions of system (1). Here the coefficients of system (1) were assumed to satisfy the following conditions in $\bar{G}$: 1) $A_k(t,x)$ with $|k| = 2b$ satisfies a Hölder condition in t with exponent $1-(r_j+1)/2b+\varepsilon$; 2) $A_k(t,x)$ have $|k|-(r_j+1)$ continuous derivatives with respect to x, Hölder in x. For $|k| < r_j + 1$, $A_k(t,x)$ are continuous and Hölder in x.

In the case of a nonconvex domain V, such a construction is no longer possible, since the p.f.m.s. studied in ([1,2]), continued into the half-space $(x,\nu(\xi)) < 0$, has, generally speaking, singularities at $t=\tau$ inside the cone

$(x-\xi,\nu(\xi)) \le |x-\xi|\cos\alpha \quad \big(\tan\alpha = (c_2/c_1)^{(2b-1)/2b};$ c_1, c_2 are constants from estimate (7) of ([1]))

$((x-\xi,\nu(\xi)) < 0)$, which may lie inside this domain. Therefore it is necessary to change the method of constructing the s.m.s. With respect to the domain V we shall assume that there exists such a sufficiently small $h > 0$ that at any point ξ of the surface S one can place the vertex of a right circular cone with axis $-\nu(\xi)$, aperture 2β ($\beta > \alpha$), and height $2h$, lying entirely outside V (the cone property). Denote by V_ξ the domain obtained from the whole space E_n by removing the right circular cone with axis $-\nu(\xi)$, vertex at the point (τ,ξ), and aperture 2β. We define the columns of the s.m.s. $E(t,\tau,x,\xi)$ as follows:

$$E_j(t,\tau,x,\xi) = G_j^*(t-\tau,x-\xi,\tau,\xi) - \\ - \int_\tau^t d\beta \int_{V_\xi} Z(t,\beta,x,y) L[G_j^*(\beta-\tau,y-\xi,\tau,\xi)]\,dy. \tag{7}$$

The matrix E thus defined has, generally speaking, singularities inside the domain V at $t=\tau$; therefore we shall construct for E a special addition which, while preserving the character of the behavior of this matrix in a neighborhood of the point $(\tau,\xi) \in \Gamma = S \times (0,T]$, will eliminate its possible singularities inside the domain at $t=\tau$. For this we shall assume that: 3) the coefficients of system (1), $A_k(t,x)$, have derivatives with respect to x of order $|k|$, Hölder in x. Define the column:

$$M_j(t,x) = \lim_{\rho\to 0} \int_0^t d\tau \int_{\sigma_h - \sigma_\rho + S_\rho} \sum_{k=1}^n B^k[Z(t,\tau,x,\xi), E_j(\tau,\xi)]\nu_k\,dS \\ = \lim_{\rho\to 0} M_j^{(\rho)}(t,x); \tag{8}$$

σ_h is the lateral surface of the cone of height h with axis $-\nu(\xi)$ and vertex at the point under consideration; σ_ρ is the part of the cone lying inside the ball of radius ρ centered at the point under consideration, and S_ρ is the surface of this ball; $B^k[Z(t,\tau,x,\xi),E_j(\tau,\xi)]$ are bilinear forms in Z and E_j and their derivatives with respect to $x_1,\dots,x_n$ up to order $2b-1$; ν_k are the direction cosines

of the normal to the surface of integration. Then, by virtue of the representation

$$E_j(t,x)=\int_0^t d\tau\int_{\sigma-\sigma_\rho+S_\rho}\sum_{k=1}^{n}B^k[Z(t,\tau,x,\xi),E_j(\tau,\xi)]\,\nu_k\,dS \tag{9}$$

and the independence of $M_j^{(\rho)}(t,x)$ of ρ,

$$M_j(t,x)=E_j(t,x)+\Delta E_j(t,x)-\int_0^t d\tau\int_{\sigma-\sigma_h}\sum_{k=1}^{h}B^k[Z(t,\tau,x,\xi),E_j(\tau,\xi)]\,\nu_k\,dS. \tag{10}$$

Remark. In the case of the first boundary-value problem, as shown in ([1]), the f.m.s. has a singularity only at the point $t=\tau$, $x=\xi$; therefore the last constructions are superfluous in this case, and the s.m.s. is constructed by formula (6). Consider problem (1)–(2) for the cylindrical domain $V\times(0,T]$ with bounded boundary S, possessing the cone property and belonging to the class $A^{(1,\lambda)}$ ([6]). By means of the potential

$$\int_0^t d\tau\int_S M(t,\tau,x,\xi)\,\mu(\tau,\xi)\,d_\xi S$$

problem (1)–(2) is reduced to a solvable system of Volterra integral equations of the second kind.

Theorem 1. *If the coefficients of system (1) satisfy conditions 1), 2), and, in the case of a nonconvex domain, in addition condition 3) with $r_j=\bar r$, the coefficients of the boundary operator are defined in a strip adjoining the surface $S\times(0,T]$ and are Hölder continuous in t and x, and $f(t,x)$ is continuous and bounded on $S\times(0,T]$, then a solution of problem (1)–(2) exists.*

2. We now pass to consideration of problem (1)–(2) for bounded noncylindrical domains G, satisfying the following conditions: a) no hyperplane $t=\tau$ is tangent to the boundary S of the domain G; b) the section G_τ of the domain G by any hyperplane $t=\tau$ satisfies the conditions indicated above for the domain V (the cone property), and the cone is the same for all sections of the domain G; c) for every point of the boundary S there exists a neighborhood S_a on S (in the aggregate of variables t,x) such that, in some coordinate system, the equation of S_a is representable in the

form $\xi_n = \varphi(\tau, \xi_1, \dots, \xi_{n-1})$, with a function φ differentiable with respect to $\xi_1, \xi_2, \dots, \xi_{n-1}$, and, moreover,

$$|\varphi(t,\xi') - \varphi(\tau,\xi')| \le C_0(t-\tau)^{\alpha_1}, \quad |\partial\varphi(t,\xi')/\partial\xi_j - \partial\varphi(\tau,\xi')/\partial\xi_j| \le C_1(t-\tau)^{\alpha_2},$$

$$\alpha_1 \ge 1/2b, \quad \alpha_2 > 0, \quad j = 1, 2, \dots, n-1;$$

C_0, C_1 are absolute constants.

Let the condition be fulfilled

$$\left|\det\left\{\int_\Gamma (B_0(\tau,\xi,\zeta+\mu\nu(\tau,\xi))A_{0+}^{-1} + (\tau,\xi;p,\zeta+\mu\nu(\tau,\xi))M_1(\mu))\,d\mu\right\}\right| \ge$$

$$\ge \delta_1\left(|\zeta|^2 + |p|^2\right)^{m/2}, \tag{11}$$

$$m = \sum_{i=1}^{bN} r_i - \frac{Nb(b-1)}{2},$$

(τ, ξ) is an arbitrary point of the boundary of the domain G; $\nu(\tau, \xi)$ is the inward normal to the boundary of the domain G_τ; ζ is any real vector orthogonal to $\nu(\tau, \xi)$; $p = a_0 + ip_1,\ a_0 > 0,\ -\infty < p_1 < \infty$; A_{0+} is the matrix defined in ([2]); Γ is a closed contour in the complex μ-plane, enclosing all μ-roots of the equation

$$\det A_{0+}(\tau, \xi, p, \zeta + \mu\nu(\tau, \xi)) = 0,$$

δ_1 is an absolute positive constant.

For simplicity, we shall carry out the further constructions in the case when the sections G_τ of the domain G are convex. Starting from the f.m.s. of problem (3)–(4), for which in the half-space $(x, \nu(\tau, \xi)) > 0$ the estimates (5) hold, we construct

columns of the matrix E in the following way:

$$E_j(t,\tau,x,\xi) = G_j^*(t-\tau, x-\xi, \tau, \xi) -$$

$$-\int_\tau^t d\beta \int_{G_\beta} Z(t,\beta,x,y) L(G_j^*(\beta-\tau, y-\xi, \tau, \xi))\,dy. \tag{12}$$

Because G is noncylindrical, the domain of integration G_β may contain points from the half-space $(y - \xi, \nu) < 0$, at which the estimates (5) are not preserved. However, by dividing the domain of integration into two parts, corresponding to the points of the half-spaces $(y - \xi, \nu) > 0$ and $(y - \xi, \nu) \le 0$, and using the Hölder continuity in t of the boundary of the domain G, one can show that for

E_j the estimates (5) hold for $m_0 = 0,\ |m| \leq 2b-1$. The solution of problem (1)–(2) for a noncylindrical domain is sought in the form

$$u(t,x) = \int_0^t d\tau \int_{S_\tau} E(t,\tau,x,\xi)\mu(\tau,\xi)\, d_\xi S;$$

S_τ is the section of the surface S by the hyperplane $t=\tau$; $\mu(\tau,\xi)$ is an unknown vector-function. With the aid of the equality

$$\lim_{x\to z\in S} B\int_0^t d\tau \int_{S_\tau} E(t,\tau,x,\xi,\tau)\mu(\tau,\xi)\, d_\xi S =$$

$$= \mu(t,z) + \int_0^t d\tau \int_{S_\tau} BE(t,\tau,z,\tau,\xi)\mu(\tau,\xi)\, d_\xi S,$$

the problem is reduced to a solvable integral equation. In the case when the sections are nonconvex, one first constructs the matrices $E(t,\tau,x,\xi)$ and $M(t,\tau,x,\xi)$ by means of a method very close to that set forth in Sec. 1, and then considers the potential

$$\int_0^t d\tau \int_{S_\tau} M(t,\tau,x,\xi)\mu(\tau,\xi)\, d_\xi S.$$

Theorem 2. *If the coefficients of the system (1) satisfy in the domain the conditions: 1) the coefficients of the system satisfy condition 1), 2) of Sec. 1, and in the case of nonconvex sections G_τ, additionally condition 3) of Sec. 1; 2) the domain G satisfies conditions a), b), c), the coefficients of the boundary operator and the function $f(t,x)$ satisfy the conditions of Theorem 1, and condition (11) is fulfilled, then a solution of problem (1)–(2) for a noncylindrical domain exists.*

3. The constructions given above make it possible to prove the solvability of boundary-value problems for certain unbounded domains.

Consider, in the space of $n+1$ dimensions $t, x_1, \dots, x_n$, the layer $\Pi_T\{0 \leq t \leq T,\ -\infty < x_s < \infty\}$ and a nonconvex set F, obtained from this layer by deleting the set $\bar{G}$, where G is a cylindrical or noncylindrical domain with the properties described in Secs. 1 and 2. The problem is considered of finding in the domain F a regular solution of the system (1), satisfying the boundary conditions (2), where the point $x \to \xi \in S$, while remaining in F. Since the special fundamental matrices of solutions needed for constructing the potentials are studied in the case of such unbounded domains in exactly the same way as in the case of bounded domains, the results of Theorems 1 and 2 extend to this case as well. If the coefficients of the senior group of the system (1) and of the boundary operator (2) are constant, then the results of Theorems 1 and 2 are also valid when the boundary is unbounded.

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Note: Figure translations are in progress. See original paper for figures.