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Abstract

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FUNDAMENTAL MATRICES OF SOLUTIONS OF PARABOLIC AND ELLIPTIC SYSTEMS WITH COEFFICIENTS SATISFYING AN INTEGRAL HÖLDER CONDITION

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Fundamental matrices of solutions (f.m.s.) of parabolic and elliptic systems have at present been constructed under the assumption that the coefficients of the system satisfy the Hölder condition ($^{1-3}$). In this note f.m.s. are constructed under weaker restrictions on the coefficients of the system: it is assumed that the coefficients of the system satisfy an integral Hölder condition (the Dini condition). In this case the f.m.s. of a parabolic system is defined as a matrix $Z(t, \tau, x, \xi)$, whose columns satisfy the system for $t > \tau$, $x \neq \xi$, analogously to the way in which the fundamental solution was defined in (^{3}). For systems of second order more precise results are obtained.

1. Construction of fundamental matrices of solutions of parabolic systems

Definition. We shall say that a function $f(x, t)$, continuous in the domain $Q = \Omega \times [t_0, T]$, where Ω is some domain in the space $(x_1, \dots, x_n)$, belongs to the class H^m if its modulus of continuity with respect to x

$$\omega(h) = \sup_{|x-\xi| \leq h,\ t_0 \leq t \leq T} |f(x,t) - f(\xi,t)|$$

has the property

$$\omega(h) \leq \prod_{k=1}^{m} \omega_k(h),$$

where $\omega_k(h)$ are continuous nonincreasing nonnegative functions such that, for some a, the integrals

$$\int_0^a \frac{\omega_k(h)}{h}\, dh$$

converge.

Consider a system parabolic in the sense of I. G. Petrovskii,

$$\frac{\partial u}{\partial t} = \sum_{|k|=2b} A_k(x,t)D^k u + \sum_{|k|\leq 2b-1} A_k(x,t)D^k u \equiv P_0(x,t,D)u + P_1(x,t,D)u = Pu, \tag{1}$$

where

$$D^k = \partial^{k_1+\dots+k_n}/\partial^{k_1}x_1 \cdots \partial^{k_n}x_n,$$

whose coefficients are defined in the strip $\Pi\{t_0 \leq t \leq T,\ -\infty < x_s < +\infty,\ s = 1,2,\dots,n\}$.

Theorem 1. If: a) the coefficients of system (1) are continuous and bounded functions of x,t in Π, and the continuity with respect to t of the coefficients of $P_0(x,t,D)$ is uniform in x; b) the coefficients of $P_0(x,t,D)$ belong to the class H^3 with function

$$\omega(h) \leq \omega_1(h)\omega_2(h) \leq \omega_3(h)\omega_4(h)\omega_2(h),$$

and the coefficients of $P_1(x,t,D)$ belong to the class H^2 with function $\tilde{\omega}(h)$ (which, for simplicity, we take to coincide with $\omega_1(h)$), then there exists an f.m.s. $Z(t,\tau,x,\xi)$ of system (1), satisfying the estimates

$$|D^m Z(t,\tau,x,\xi)| \leq C_m(t-\tau)^{-|m|/2b}\left\{(t-\tau)^{-n/2b} + \omega_2[(t-\tau)^\alpha]\,\omega_3(|x-\xi|^\alpha)\,|x-\xi|^{-n}\right\}\exp\{-c\rho\}. \tag{2}$$

If, in addition to conditions a) and b), the following condition is satisfied: c) the coefficients $P_0(x,t,D)$ belong to the class H^4 $(\omega_3(h) \leqslant \omega_5(h)\omega_6(h))$, then

$$\begin{aligned}|\Delta_h D^m Z(t,\tau,x,\xi)| \leqslant \widetilde{C}_m \omega_3(|h|^\alpha)(t-\tau)^{-(|m|+\alpha)/2b}\{(t-\tau)^{-n/2b}+\\ +\omega_5(|x-\xi|^\alpha)|x-\xi|^{-n} + |x+h-\xi|^{-n}\omega_5(|x+h-\xi|^\alpha)\}\exp(-c\rho)\end{aligned} \tag{3}$$

for $|h| \leqslant a(t-\tau)^{1/2b}$, where $|m| \leqslant 2b$,

$$\rho = \sum_{s=1}^{n} |x_s - \xi_s|^q (t-\tau)^{-1/(2b-1)},$$

$0 < \alpha < 1/2b$, $q = 2b/(2b-1)$, $\Delta_h f(x) = f(x+h) - f(x)$, $c, C_m, \widetilde{C}_m$ are positive constants, and a is an arbitrary positive number.

Proof. Let the system

$$\partial u/\partial t = P(x,t,D)u - Au \tag{4}$$

have the f.m.s. $\widetilde{Z}(t,\tau,x,\xi)$. Then $Z(t,\tau,x,\xi) = \widetilde{Z}(t,\tau,x,\xi)\exp\{A(t-\tau)\}$ is the f.m.s. of system (1). Thus it suffices to construct $\widetilde{Z}(t,\tau,x,\xi)$. Denote the Green matrix of the auxiliary system $\partial u/\partial t = P_0(y,t,D)u - Au$ by $G(t,\tau,x-\xi,y)$. Following Levi, we shall seek $\widetilde{Z}(t,\tau,x,\xi)$ in the form

$$\begin{aligned}\widetilde{Z}(t,\tau,x,\xi) = G(t,\tau,x-\xi,\xi) + \int_\tau^t d\beta \int G(t,\beta,x-y,y)\varphi(\beta,\tau,y,\xi)\,dy\\ \equiv G(t,\tau,x-\xi,\xi) + W(t,\tau,x,\xi).\end{aligned} \tag{5}$$

Suppose that for the function $\varphi(t,\tau,x,\xi)$ the estimates

$$|\varphi(t,\tau,x,\xi)| \leqslant C(t-\tau)^{-1}\omega_2[(t-\tau)^\alpha]|x-\xi|^{-n}\omega_1(|x-\xi|^\alpha)\exp\{-c\rho\}, \tag{6}$$

$$\begin{aligned}|\Delta_h\varphi(t,\tau,x,\xi)| \leqslant C(t-\tau)^{-1}\omega_2[(t-\tau)^\alpha]\omega_1(|h|^\alpha)\left\{(t-\tau)^{-n/2b}+\right.\\ \left.+\frac{\omega_3(|x-\xi|^\alpha)}{|x-\xi|^n} + \frac{\omega_3(|x+h-\xi|^\alpha)}{|x+h-\xi|^n}\right\}e^{-c\rho}\end{aligned} \tag{7}$$

hold for $|h| \leqslant a(t-\tau)^{1/2b}$. These a priori assumptions are proved below. Then, with the help of the arguments of (1^a) and of the easily proved inequality

$$\int \exp\left[-\left(\frac{|x-y|}{h^{1/2b}}\right)^q\right] h^{-1/2b}\frac{\omega_1(|z-y|)\omega_2(|y-\xi|)}{|y-\xi|^n}\,dy\bigg|_{z=x} \leqslant C\frac{\omega_1(2|x-\xi|^\alpha)}{|x-\xi|^n} \tag{8}$$

(this inequality remains valid also for $z=\xi$), it is shown that the function $W(t,\tau,x,\xi)$, for $t>\tau,\ x\neq\xi$, has all the derivatives entering system (1). In order that $\widetilde{Z}(t,\tau,x,\xi)$, as a function of x,t for $t>\tau,\ x\neq\xi$, be a solution of system (1), it is necessary that $\varphi(t,\tau,x,\xi)$ satisfy the integral equation

$$\varphi(t,\tau,x,\xi) = K(t,\tau,x,\xi) + \int_\tau^t d\beta\int K(t,\beta,x,y)\varphi(\beta,\tau,y,\xi)\,dy, \tag{9}$$

where

$$K(t,\tau,x,\xi) \equiv [P_0(x,t,D) - P_0(\xi,t,D)]G + P_1(x,t,D)G.$$

Thus,

$$\varphi(t, \tau, x, \xi) = \sum_{m=1}^{\infty} K_m(t, \tau, x, \xi). \tag{10}$$

By virtue of the hypotheses of the theorem, the estimates of the Green matrix (^{1}), and inequality (8), the following estimate of the iterated kernels is established:

$$|K_m(t, \tau, x, \xi)| \leqslant C_0^m F^{m-1}(T)(t-\tau)^{-1}\omega_2[(t-\tau)^\alpha]|x-\xi|^{-n}\omega_1(|x- \\ -\xi|^\alpha) \exp\left\{-c\rho - \frac{A}{2}(t-\tau)\right\}, \tag{11}$$

where

$$F(T) = C' \int_0^{T^\alpha} \frac{\omega_2(x)}{x} \exp\left[-\frac{A}{2} x^{1/\alpha}\right] dx, \quad C' > 0.$$

Hence,

$$|\varphi(t, \tau, x, \xi)| \leq \frac{\omega_2[(t-\tau)^\alpha]}{t-\tau} \frac{\omega_1(|x-\xi|^\alpha)}{|x-\xi|^n} e^{-c\rho} \sum_{m=1}^{\infty} C_0^m F^{m-1}(T).$$

Choosing the arbitrarily taken constant A so large that

$$\sum_{m=1}^{\infty} C_0^m F^{m-1}(T) < +\infty,$$

we obtain that, for $|x-\xi| \geq \delta$, $t-\tau \geq \delta$, where δ is a sufficiently small positive number, the series (10) converges uniformly and absolutely, and for its sum the estimate (6) is valid. In establishing estimate (7) the following cases are distinguished: 1) $|x-\xi| \leq 2|h|^\alpha$, 2) $|x-\xi| < 2|h|^\alpha$, $0 < \alpha < 1/2b$; the estimate of the Green matrix is also used. The proof of estimate (8) is based on the method of (^{1b}), by means of which the Hölder property of the fundamental matrix of solutions was established for systems with Hölder coefficients.

Remark. For systems of second order ($b = 1$) the estimate of the norms of the repeated kernels can be carried out without using estimate (8), by means of the convolution formula for the fundamental solution of the heat equation. This makes it possible, under weaker restrictions on the function $\omega(h)$, to construct the fundamental matrix of solutions and to obtain for it the usual estimates (^{1a}).

2. Construction of the fundamental matrix of solutions of an elliptic system which is the right-hand side of a parabolic system in the sense of I. G. Petrovskii. Consider the elliptic system

$$P(x,D)u = P_0(x,D)u + P_1(x,D)u = \sum_{|k|=2b} A_k(x)D^k u + \sum_{|k|\le 2b-1} A_k(x)D^k u. \tag{12}$$

Theorem 2. *If: 1) the coefficients of system (12) are defined, continuous, and bounded in a domain Ω sufficiently small in diameter; 2) the coefficients of $P_0(x,D)$ belong to the class H^2, and the coefficients of $P_1(x,D)$ to the class H^1, then in every subdomain Ω_1 strictly contained in Ω, there exists a fundamental matrix of solutions of system (12) satisfying the estimates*

$$|D^m\varphi(x,\xi)| \le \begin{cases} C, & n+|m|\le 2b-1, \\ C\ln\dfrac{1}{|x-\xi|}+C_1, & n+|m|=2b, \\ Cm|x-\xi|^{-n-|m|+2b}, & n+|m|>2b, \end{cases} \qquad x\in\Omega_1,\ \xi\in\Omega. \tag{13}$$

We give the scheme of proof of this theorem. Denote by $G_0(t-\tau, x-\xi, y)$ the Green matrix of the parabolic system $du/dt = P_0(y,D)u$. The fundamental matrix $\varphi_0(x-\xi, y)$ of the elliptic system $P_0(y,D)u=0$ is defined by formula (31) of paper (^{1b}). Suppose a priori that the function $f(x,\xi)$ satisfies the inequalities

$$|f(x,\xi)| \le C|x-\xi|^{-n}\omega_2(|x-\xi|^\alpha), \tag{14}$$

$$|\Delta_h f(x,\xi)| \le C\omega_1(|h|^\alpha)\left\{\frac{\omega_2(|x-\xi|^\alpha)}{|x-\xi|^n} + \frac{\omega_2(|x+h-\xi|^\alpha)}{|x+h-\xi|^n}\right\}. \tag{15}$$

Applying to the function

$$\int_0^t d\tau \int_\Omega [G_0(t-\tau, x-y, y) - \mathcal{P}_{2b-n}(t-\tau, x-y, a)] f(y,\xi)\, dy,$$

$x\ne\xi,\ x\in\Omega_1$, the operator $P-\partial/\partial t$ and then passing to the limit as $t\to\infty$, taking account of (31) from (^{1b}), we obtain the formula

$$P(x,D)\int_\Omega \varphi_0(x-y,y) f(y,\xi)\, dy =$$

$$= -f(x,\xi) + \int_\Omega [P(x,D)-P_0(\xi,D)]\varphi_0(x-y,y) f(y,\xi)\, dy. \tag{16}$$

Seeking the f.m.s. of system (12) in the form

$$\varphi(x,\xi) = \varphi_0(x-\xi,\xi) + \int_\Omega \varphi_0(x-y,y) f(y,\xi)\, dy, \tag{17}$$

we arrive at an integral equation for determining $f(x,\xi)$:

$$f(x,\xi) = K(x,\xi) + \int_\Omega K(x,y) f(y,\xi)\, dy, \tag{18}$$

whose kernel $K(x,\xi) = [P(x,D) - P_0(\xi,D)]\varphi_0(x-\xi,\xi)$, by virtue of the conditions of the theorem and the estimates for $\varphi_0(x-\xi,\xi)$ (1), for $|x-\xi| \leqslant 1$, has the estimate

$$|K(x,\xi)| \leqslant C_0 \omega(|x-y|)/|x-y|^n. \tag{19}$$

With the aid of the inequality

$$\int_\Omega \frac{\omega_1(|x-y|)\omega_2(|x-y|)\omega(|y-\xi|^\alpha)}{|x-y|^n\, |y-\xi|^n}\, dy \leqslant C \frac{\omega_2(2|x-\xi|^\alpha)}{|x-\xi|^n} \tag{20}$$

for equation (18) "in the small" the convergence of the method of successive approximations is established, and then the a priori assumptions are proved. The estimates (13) are established with the aid of (14), (15), (20) and the integral Hölder continuity of $\varphi_0(x-\xi,y)$ with respect to the argument y.

3. The Cauchy problem. The constructed f.m.s. of the parabolic system (1) makes it possible to consider the question of correctness classes for the Cauchy problem in the spaces of rapidly increasing functions $L_{p,k(t),N}$, introduced in (1^a). Consider the Cauchy problem for the nonhomogeneous system

$$\partial u/\partial t = P(x,t,D)u + f(x,t), \qquad u|_{t=t_0} = \varphi(x). \tag{21}$$

Theorem 3. 1) *If for the coefficients of system (1) conditions a) and b) are fulfilled, then the uniqueness class of the solution of the Cauchy problem for system (1) is the collection* $L^{(2b,\omega)}_{p,k(t),N}$ *of functions* $u(x,t)$, *whose derivatives with respect to* x *up to order* $2b$ *inclusive for* $t > t_0$ *belong to the space* $L_{p,k(t)N}$ *and, in each finite domain* $D_{t_0,a}\{|x| \leqslant a,\ t_0 < t \leqslant t_1\}$, *satisfy the integral Hölder condition.* 2) *If* $f(x,t)$ *in each domain*

$$D_{t_0,a}$$

belongs to the class H^1 *and*

$$\|f(x,t)\|_{M_p} = \int_{t_0}^{t_1} \|f(x,t)\|_{L_{p,k(t),N}}\, dt < +\infty,$$

then

$$u(x,t) = \int Z(t, t_0, x, \xi)\varphi(\xi)\, d\xi + \int_{t_0}^{t_1} d\tau \int Z(t, \tau, x, \xi) f(\xi, \tau)\, d\xi \qquad (22)$$

is a solution of problem (21) in the strip $(t_0, t_1]$*, satisfying the estimate*

$$\|u(x,t)\|_{L_{p,k(t),N}} \leqslant C(\|\varphi\|_{L_{p,k(t_0),N}} + \|f(x,t)\|_{M_p}). \qquad (23)$$

3) *If* $\|f(x,t)\|_{M_p} < +\infty$ *and* $f(x,t)$ *in* $D_{t_0,a}$ *belongs to the class* H^2*, then the solution* $u(x,t)$*, defined by formula (22), belongs to the uniqueness class.*

The proof of assertion 1) is analogous to that which is carried out for systems with Hölder coefficients in (4). Theorem 3 establishes the correct solvability of the Cauchy problem in the space $L^{(2b,\omega)}_{p,k(t),N}$.

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Note: Figure translations are in progress. See original paper for figures.

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