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Abstract

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PARTITION OF THE SET OF POISSON-STABLE MOTIONS INTO INVARIANT CLASSES

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In the present note a partition ([1]) is constructed of the set of all Poisson-stable motions of dynamical systems ([2,3]), induced (in the sense of Definition 1) by the known classes of these motions. It is shown that motions of all classes of the constructed partition exist also in linear dynamical systems ([4,5]). Moreover, Theorems 3-5 establish that the linear dynamical system given in ([4]) is "universal."

1°. Consider the set Π of all Poisson-stable motions of dynamical systems ([2,3]). In it one distinguishes the classes of stationary, periodic, almost periodic, uniformly Poisson-stable, recurrent, almost recurrent ([6]), and pseudo-recurrent ([7,8]) motions. We shall call the enumerated classes the **basic classes of Poisson-stable motions**, or, more briefly, the **basic classes** (since in what follows only Poisson-stable motions will be discussed).

It is known that the basic classes are invariant, i.e. if a motion $f(p,t)$ belongs to some basic class, then the motion $f(q,t)$ also belongs to it, whatever the point $q \in f(p,I)$. In addition, each of the basic classes is nonempty and no two of them coincide. However, there exist Poisson-stable motions that belong simultaneously to several basic classes, and also motions that belong to none of them. In this connection it is of interest to construct a partition of the set Π containing a minimal number of classes, in such a way that every basic class is a union of some number of classes of this partition. Such a partition is constructed below.

Definition 1. Let $\{A_i\}$ be a given system of proper subsets of some set A. Denote by π_i the partition of the set A consisting of A_i and $A \setminus A_i$. The intersection of all partitions π_i ([9]) will be called the **partition of the set A induced by the subsets $\{A_i\}$**.

Consider the following invariant sets of motions: Π_1 is the set of all stationary motions; Π_2 is the set of all periodic motions that are not stationary; Π_3 is the set of all almost periodic motions that are not periodic (and hence, in particular, not stationary); Π_4 is the set of all recurrent motions that are uniformly Poisson-stable but not almost periodic; Π_5 is the set of all recurrent motions that are not uniformly Poisson-stable; Π_6 is the set of all almost recurrent motions that are uniformly Poisson-stable but not recurrent; Π_7 is the set of all almost recurrent motions that are pseudo-recurrent but not recurrent and not uniformly Poisson-stable; Π_8 is the set of all almost recurrent motions that are not pseudo-recurrent; Π_9 is the set of all uniformly Poisson-stable motions that are not almost recurrent; Π_{10} is the set of all pseudo-recurrent motions that are neither almost recurrent nor uniformly Poisson-stable; Π_{11} is the set of all Poisson-stable motions that are neither almost recurrent nor pseudo-recurrent.

The known connection between the basic classes of motions makes it possible to establish that if each of the enumerated sets is nonempty, then they constitute a partition of the set Π induced by the basic classes. On the other hand—

on the other hand, the examples of motions given in item 2^0 of the present note show that the sets $\Pi_6 - \Pi_8$ are nonempty. Examples of motions belonging to the remaining sets Π_i $(i = 1, 2, 3, 4, 5, 9, 10, 11)$ can be found in ([2,3,6–8]). Thus the following holds.

Theorem 1. *The partition of the set of all Poisson stable motions, induced by the principal classes, consists of the sets* Π_i $(i = 1, 2, \dots, 11)$*, each of which is an invariant class of motions.*

In what follows, the sets Π_i $(i = 1, 2, \dots, 11)$ will be called the **component classes of Poisson stable motions**.

2^0. Let us give several examples of Poisson stable motions. These examples are constructed in the dynamical system of M. V. Bebutov ([6]), which we shall denote by (R_u, β). Throughout we shall use the following notation: N is the set of all natural numbers, E is the set of all integers, I is the set of all real numbers.

Consider the following three sequences:

$$a_n = 4(n+2)\ (n \in N); \qquad l_0 = 0,\ l_1 = 1,\ l_{n+1} = a_{nl} n\ (n \in N);$$

$$\tau_n = \sum_{s=0}^{n-1} l_s\ (n \in N).$$

Let $n \in N$, $i \in E$, and $x \in I$. Introduce the following notation: $\sigma(i, n)$ is the interval $((2i-1)l_n + \tau_n, (2i+1)l_n + \tau_n]$; $i_n(x)$ is the integer such that $x \in \sigma(i_n(x), n)$; $j_n(x)$ is the integer equal to one if $n = 1$, and equal to the difference $i_{n-1}(x) - a_{n-1} i_n(x)$ if $n \geqslant 2$; $r(x)$ is the least natural number such that $j_{r(x)+1}(x)$ is an even number. Define on I a continuous function $p(x)$, putting $p(x) = \varphi(x)\psi(x)$, where

$$\varphi(x) = \prod_{n=1}^{r(x)} \frac{2n + 1 - |j_n(x)|}{2n};$$

$$\psi(x) = \begin{cases} 0, & \text{if } x - 2i_1(x) \in \left[-1, -\dfrac{1}{r(x)}\right] \cup \left[\dfrac{1}{r(x)}, 1\right], \\ 1 - |x - 2i_1(x)|\, r(x), & \text{if } x - 2i_1(x) \in \left[-\dfrac{1}{r(x)}, \dfrac{1}{r(x)}\right]. \end{cases}$$

Let us list some properties of the function $p(x)$: a) the function $p(x)$ is continuous and $0 \leqslant p(x) \leqslant 1$ on I; b) the function $p(x)$ is not uniformly continuous on I; c) for every $\varepsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $|p(x+\tau) - p(x)| < \varepsilon$ for all $|x| \leqslant 1/\varepsilon$; d) the function $p(x)$ is pseudo-periodic in the sense of Bohr.

In the dynamical system (R_u, β) consider the motion $\beta(p,t)$ determined by the function $p(x)$. From property c) it follows that the motion $\beta(p,t)$ is almost recurrent, and from property d) that it is uniformly Poisson stable. However, by Birkhoff' s theorem, the motion under consideration is not recurrent, since its trajectory is not a compact set. The latter follows from property b).

Thus, the motion $\beta(p,t)$ is almost recurrent and uniformly Poisson stable, but not recurrent.

If in the example given above we slightly change the construction of the function $p(x)$, then one can ensure that it is not pseudo-periodic in the sense of Bohr, but has a weaker property, which consists in the following:

d$'$) for every $\varepsilon > 0$ there exists $L \geqslant 1$ such that, whatever $t \in I$ may be, there is a $\tau \in [1, L]$ for which $|p(x+t+\tau) - p(x+t)| < \varepsilon$ for all $|x| \leqslant 1/\varepsilon$.

For this it is sufficient to take $\sigma(i,n) \equiv ((2i-1)l_n, (2i+1)l_n]$, and define the sequence $\{a_n\}$ by the equality $a_n = 4n+7$. Under these changes properties a)–c) of the function $p(x)$ are preserved. Therefore the motion $\beta(p,t)$ will remain almost recurrent and nonrecurrent. However, it already

will not be uniformly Poisson-stable, but will be pseudorecurrent. The latter follows from property d$'$).

Thus, there exist almost recurrent motions that are pseudorecurrent, but not recurrent and not uniformly Poisson-stable.

Let us now give an example of an almost recurrent motion that is not pseudorecurrent. Let $x \in I$. By $\gamma(x)$ we shall denote an odd number γ such that $x \in [\gamma-1, \gamma+1]$, and by $\mu(x)$ a natural number μ such that $\gamma(x) = 3^{\mu+1}(6k \pm 1)$, where $k \in E$.

Define on I a function $p(x)$, putting

$$p(x) = \begin{cases} 0, & \text{if } x - \gamma(x) \in [-1, -2^{1-\mu(x)}] \cup [2^{1-\mu(x)}, 1], \\ 1 - |x| 2^{\mu(x)-1}, & \text{if } x - \gamma(x) \in [-2^{1-\mu(x)}, 2^{1-\mu(x)}]. \end{cases}$$

The function just defined has the following properties:
a) the function $p(x)$ is continuous and $0 \leq p(x) \leq 1$ on I; b) for every $\varepsilon > 0$ there exists a relatively dense set T such that, if $\tau \in T$, then $p(x+\tau) = p(x)$ for all $|x| \leq 1/\varepsilon$; c) for any $n \in N$ and $\tau \in [1, 3^n]$

$$\sup_{-1 \leq x \leq 1} |p(x + 3^n + \tau) - p(x + 3^n)| \geq \frac{1}{2}.$$

It follows from the properties of the function $p(x)$ that in the dynamical system (R_u, β) the function $p(x)$ defines an almost recurrent motion that is not pseudorecurrent.

We note that the closure of the trajectory of this motion is a minimal set that is not pseudominimal. The question of the existence of sets of this kind was posed in ([3]) (p. 151, item 6).

3°. In the definition and study of linear dynamical systems ([4,5]), the question was posed of whether there exist in them all the variety of Poisson-stable motions that has been found in arbitrary dynamical systems. The answer to it follows from the following considerations.

Definition 2. It is said that a dynamical system (R_1, f_1) is mapped into a dynamical system (R_2, f_2) if there exists a mapping h of the space R_1 into R_2 such that

$$h(f_1(p,t)) = f_2(h(p),t)$$

for any point $p \in R_1$ and any $t \in I$. In this case, if h is a homeomorphism (equimorphism*), then it is said that (R_1, f_1) is mapped homeomorphically (equimorphically) into (R_2, f_2).

Theorem 2. The constituent classes of Poisson-stable motions are invariant with respect to equimorphic mappings of dynamical systems.

The validity of this theorem is easy to establish. It follows from it that the constituent classes of Poisson-stable motions are invariant with respect to homeomorphic mappings of compact dynamical systems.

In what follows, without special mention, we shall use the fact that if (R, f) is a dynamical system and M $(M \subseteq R)$ is an invariant set, then (M, f) is also a dynamical system.

Let (B_u, λ) be the linear dynamical system that was presented as an example in ([4]), with $\alpha(x) \equiv |x|$.

Theorem 3. If M is an invariant set of uniformly bounded functions in (R_u, β), then the dynamical system (M, β) can be equimorphically mapped into the linear dynamical system (B_u, λ).

Proof. It is obvious that $M \subset B_u$. The set M, considered as an invariant set of the space B_u, will be denoted by M'. Define a mapping h of the set M onto M' by putting $h(p) = p$ for all $p \in M$. It is clear that h is one-to-one and

$$h(\beta(p,t)) = \lambda(h(p),t)$$

for all $p \in M$ and $t \in I$. We shall show that h is uniformly continuous in both directions.

Let $\varepsilon > 0$. Choose δ so that $0 < \delta < \varepsilon$ and $2ce^{-1/\delta} < \varepsilon$, where the number c

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* An equimorphism is a one-to-one mapping that is uniformly continuous in both directions ([10]).

found from the condition of uniform boundedness of the functions of the set M. Choose $p, q \in M$ so that $\rho(p,q) \leqslant \delta$ (in the metric of the space R_u). Then

$$\sup_{|x| \leqslant 1/\delta} |p(x) - q(x)| \leqslant \delta,$$

whence

$$\sup_{|x| \leqslant 1/\delta} |p(x) - q(x)| e^{-|x|} \leqslant \delta < \varepsilon.$$

On the other hand,

$$\sup_{|x| > 1/\delta} |p(x) - q(x)| e^{-|x|} \leqslant 2ce^{-1/\delta} < \varepsilon.$$

Consequently,

$$\sup_{x \in I} |p(x) - q(x)| e^{-|x|} < \varepsilon,$$

i.e. $\|p - q\| < \varepsilon$ (see (4)).

Let now $p, q \in M'$, and let $\varepsilon > 0$ be arbitrary. Suppose that $\|p - q\| < \varepsilon e^{-1/\varepsilon}$. This means that

$$\sup_{x \in I} |p(x) - q(x)| e^{-|x|} < \varepsilon e^{-1/\varepsilon}.$$

It follows that for every $x \in [-1/\varepsilon, 1/\varepsilon]$ the inequality $|p(x) - q(x)| < \varepsilon$ holds, i.e. $\rho(p,q) < \varepsilon$.

Theorem 4. *Every compact dynamical system (R, f) having no more than one rest point can be equimorphically mapped into the linear dynamical system (B_u, λ).*

Proof. According to a theorem of M. V. Bebutov (see (6), p. 28, Theorem 6), a compact dynamical system (R, f) having no more than one rest point can be homeomorphically mapped into (R_u, β); moreover, as is easily seen from the proof of the theorem just mentioned, the compact set R is mapped onto a certain invariant set M of uniformly bounded functions. According to Theorem 3, (M, β) can be equimorphically mapped into (B_u, λ).

Corollary. *In the linear dynamical system* (B_u, λ) *there exist Poisson-stable motions of all constituent classes.*

Indeed, whichever class Π_i $(i = 1, 2, \dots, 10)$ one chooses, there is a function $p_i(x)$, bounded on I, defining in (R_u, β) a motion $\beta(p_i, t)$ that belongs to the given class Π_i (see $(3, 6\text{–}8)$ and item 2^0 of the present note). The trajectory $T_i = \beta(p_i, I)$ is an invariant set of functions uniformly bounded on I. By Theorem 3, there is an equimorphism h mapping (T_i, β) into (B_u, λ). Hence, by Theorem 2, in (B_u, λ) the motion $\lambda(h(p_i), t)$ belongs to the given class Π_i.

Using Theorems 2 and 4 and taking into account that in a compact dynamical system with one rest point, defined on a torus (see $(3, 7, 8)$), there exists a motion belonging to the class Π_{11}, by analogous reasoning one can verify that a motion of this class also exists in (B_u, λ).

From the theorem of M. V. Bebutov (see (6), p. 14, Theorem 5) and Theorem 4 it follows

Theorem 5. *Every dynamical system situated in a locally compact space with a countable base can be mapped into the linear dynamical system* (B_u, λ)*, and in such a way that on the set of all points that are not rest points the mapping will be topological.*

Theorems 3–5 show that dynamical systems from a very broad class can be topologically (homeomorphically) mapped into (B_u, λ). For these systems (B_u, λ) is universal.

In conclusion, I consider it my pleasant duty to express gratitude to K. S. Sibirsky, under whose supervision the present work was carried out.

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CITED LITERATURE

1. N. Bourbaki, *Théorie des ensembles*, Paris, 1954.

2. V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, Moscow–Leningrad, 1949.

3. V. V. Nemytskii, *Uspekhi Mat. Nauk*, **4**, 6 (1949).

4. F. A. Sholokhovich, *Izv. Vyssh. Uchebn. Zaved., Matematika*, 1 (1957).

5. G. N. Milshtein, F. A. Sholokhovich, *Sibirsk. Mat. Zh.*, **2**, 4 (1961).

6. M. V. Bebutov, *Byull. MGU*, **2**, 5 (1941).

7. B. A. Shcherbakov, *DAN*, **146**, No. 2 (1962).

8. B. A. Shcherbakov, *Izv. AN MSSR*, 1 (1963).

9. A. G. Kurosh, *Lectures on General Algebra*, Moscow, 1962.

10. V. A. Efremovich, *DAN*, **76**, No. 3 (1951).

Note: Figure translations are in progress. See original paper for figures.