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Abstract

Full Text

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ON A WIENER-HOPF EQUATION OF THE FIRST KIND

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We shall be concerned with an equation of the form

$$A\varphi \equiv \int_0^\infty c(x-u)\varphi(u)\,du = f(x), \qquad x > 0. \tag{1}$$

An extensive class of applications and a number of methods for solving this equation were first described in detail in ([1]). The results of ([1]) have been included in many modern manuals on the theory of automatic control (see, for example, ([2–5])). However, in the cited work only particular methods for solving equation (1) are given, not connected with the achievements of the theory of boundary-value problems and singular integral equations ([6,7]). This connection was first noted in ([8,9]). Nevertheless, in ([10]) it is emphasized that the theory of equation (1) is still far from complete. Later, in the note ([11]), (1) is reduced to a special case of the Riemann boundary-value problem* and, on this basis, a complete investigation of the conditions for its solvability is given and all its solutions are constructed in the class of summable functions under certain restrictions on $c(x)$ and $f(x)$. It turns out, however, that the solvability conditions for equation (1) are often quite stringent.

Here it will be shown that one can dispense with the indicated solvability conditions by enlarging the class of admissible solutions, and that this enlargement is practically expedient.

Consideration of problems in the theory of automatic control ([2–5]) suggests that a fully justified class of admissible solutions of equation (1) is the class of generalized functions of the form**

$$\varphi(x) = \sum_{j=0}^{r}\sum_{\nu=0}^{s} d_{j\nu}\delta^{(j)}(x - x_{j\nu}) + \varphi_1(x) \equiv \varphi[\varphi_1(x);\, d_{00}, d_{10}, \dots, d_{rs}], \tag{2}$$

where $d_{j\nu}$ are arbitrary constants, $\varphi_1(x) \in L_2(0,\infty)$, $x_{0\nu} \in [0,\infty)$, $x_{j\nu} \in (0,\infty)$, $j > 0$, and $\delta^{(j)}(x-x_{j\nu})$ is the derivative of order j of the Dirac function $\delta(x-x_{j\nu})$, defined by the relation

$$\int_a^b f(x)\delta^{(j)}(x-x_{j\nu})\,dx = \lim_{\alpha\to 0}\int_a^b f(x)\delta_x^{(j)}(x-x_{j\nu},\alpha)\,dx,$$

$$\delta(x,\alpha) = \frac{\alpha}{\pi(x^2+\alpha^2)}, \qquad a,b\in[-\infty,\infty], \tag{3}$$

for any function $f(x)$ given on the real axis. Substitute (2) into (1). As a result, assuming that $c(x)$ has s derivatives on the real axis, we obtain

$$\widetilde{A}\varphi \equiv \int_0^\infty c(x-u)\varphi_1(u)\,du + \sum_{j=0}^{r}\sum_{\nu=0}^{s}(-1)^j d_{j\nu}c^{(j)}(x-x_{j\nu}) = f(x), \qquad x>0, \tag{4}$$

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* It can be shown that (1) is essentially a particular case of the equations considered in ([11]).

** For other problems, but approximately for the same purposes in spirit, a similar class ($r=0$) is introduced and used in ([12]). Under certain restrictions on r and $s=0$, essentially the same class is introduced in ([13]), but the results of its use are fundamentally different from ours. The note ([14]) is also close in idea.

whence

$$A\varphi_1 = \int_0^\infty c(x-u)\varphi_1(u)\,du = f_1(x) \equiv f(x) + \sum_{j=0}^{r}\sum_{\nu=0}^{s}(-1)^j d_{j\nu}c^{(j)}(x-x_{j\nu}),$$
$$x>0. \tag{5}$$

Solving this last equation as indicated in (1), it is not difficult to show that, for sufficiently large r or s, by means of a quite definite choice of some of the constants $d_{j\nu}$, one can get rid of all the solvability conditions for (1) derived in (1).

To explain the practical expediency of the class (2), consider the following problem (1). Given: $x(t)$ and $y(t)$–records of realizations of stationary random processes, respectively at the input and output of the system sought; more precisely, $y(t)$ is the desired realization specified in advance.* It is required to determine the impulse transition function $k(t)$ that gives the least value to the mathematical expectation

$$E = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left| y(t) - \int_0^\infty x(t-\theta)k(\theta)\,d\theta \right|^2 dt. \tag{6}$$

It is known (1,2) that in order for $k(\theta)$ $(k(\theta)=0,\ \theta<0)$ to minimize (6), it is necessary and sufficient that it be a solution of the equation

$$Bk \equiv \int_0^\infty R_{xx}(u-\theta)k(\theta)\,d\theta = R_{yx}(u), \qquad u>0, \tag{7}$$

where R_{xx} and R_{yx} are, respectively, the auto- and cross-correlation functions of the signals x and y.** If $R_{xx}(u) \in L_1(-\infty,\infty)$, then, on the basis of the properties of the Fourier transform and the properties of the autocorrelation function, it is not difficult to show that B is a bounded $\left(\|B\| \leqslant \int_{-\infty}^{\infty} |R_{xx}(u)|\,du\right)$ positive definite operator in $L_2(0,\infty)$ and, therefore, equation (7) can have only a unique solution (in $L_2(0,\infty)$). However, it may happen that equation (7) is not solvable in $L_2(0,\infty)$ and is solvable in the class (2). The practical expediency of an extension of the form (2) follows directly from the fact that the terms $d_{0\nu}\delta(t-t_{0\nu})$ in $k(t)$ can easily be realized in practice by amplifier elements and elements with pure delay $t_{0\nu}$, while the terms $\delta^{(j)}(t)$ correspond to differentiating elements of order j.***

By analogy with (4), introduce the equation

$$\widetilde{B}k \equiv \int_0^\infty R_{xx}(u-\theta)k_1(\theta)\,d\theta + \sum_{j=0}^{r}\sum_{\nu=0}^{s}(-1)^j d_{j\nu} R_{xx}^{(j)}(u-t_{j\nu}) = R_{yx}(u), \qquad u>0. \tag{8}$$

We shall regard (8) as a linear equation in the Hilbert space $\widetilde{L}_2(0,\infty)$ of elements $k\,[k_1(t);\ d_{00}, d_{10}, \dots, d_{rs}]$:

$$(k^{(1)}, k^{(2)})_{\widetilde{L}2} = (k_1^{(1)}, k_1^{(2)})_{L_2} + \sum_{j=0}^{r}\sum_{\nu=0}^{s} d_{j\nu}^{(1)} d_{j\nu}^{(2)}. \tag{9}$$

* Without loss of generality, we assume that the mean values of $x(t)$ and $y(t)$ are zero.

** It is not difficult to prove that this property is also valid in the class (2).

*** If the relative smoothness of $x(t)$, besides continuity, cannot be guaranteed, then introducing derivatives

It is clear that $\widetilde{B}$ is no longer, generally speaking, a positive definite operator, and therefore equation (8) need not have a unique solution.

It is easy to see that all solutions of (8) deliver one and the same least value (6). However, if in solving equation (7) in the class (2) some of the $d_{j\nu}$ remain arbitrary, then they may be disposed of so as to satisfy certain additional requirements imposed on $k(t)$.

Of great interest is the investigation of the limiting case $r = 0$, $s = \infty$. From Wiener' s approximation theorem ([14]) it follows that if the Fourier transform S_{xx} of the function $R_{xx} \in L(0, \infty)$ is different from zero everywhere on every finite part of the real axis, then the closure of the linear manifold

$$\sum_{\nu=0}^{s} d_{0\nu} R_{xx}(u - t_{0\nu})$$

coincides with $L(0, \infty)$. This means that, in the indicated case, one of the approximate solutions of equation (7), with any prescribed accuracy of approximation (in $L(0, \infty)$), can be found for sufficiently large s ($r = 0$) in the class (2), taking $k_1(t) \equiv 0$.

Let us illustrate all that has been set forth by a simple example. Rewriting equation (7) in the form

$$R_{yx}(u) - \int_0^\infty R_{xx}(u-\theta)\, k_+(\theta)\, d\theta = \begin{cases} 0, & u \geq 0, \\ k_-(u), & u < 0, \end{cases} \tag{10}$$

and applying the Fourier transform to it, we obtain the Riemann boundary-value problem

$$S_{yx}(i\omega) - S_{xx}(i\omega)\Phi^+(i\omega) = \Phi^-(i\omega). \tag{11}$$

Let $S_{xx} = \dfrac{1}{1+\omega^2}$ and $S_{yx} = \dfrac{e^{-i\omega\tau_0}}{1+\omega^2}$. This corresponds to the special case of the problem formulated above ([1]), when $x(t)$ has the correlation function $R_{xx} = \frac{1}{2}e^{-|u|}$, and $y(t) = x(t + \tau_0)$, and therefore $R_{yx} = \frac{1}{2}e^{-|u+\tau_0|}$. In the present example, multiplying both sides of equality (11) by $\omega - i$ and observing that (see ([6], [7]))

$$\left(\frac{e^{-i\omega\tau_0}}{\omega + i}\right)^- = \frac{e^{-\tau_0} - e^{-i\omega\tau_0}}{\omega + i}, \qquad \left(\frac{e^{-i\omega\tau_0}}{\omega + i}\right)^+ = \frac{e^{-\tau_0}}{\omega + i}, \tag{12}$$

we rewrite (11) as

$$\frac{e^{-\tau_0}}{\omega + i} - \frac{\Phi^+}{\omega + i} = \Phi^-(\omega - i) - \frac{e^{-i\omega\tau_0} - e^{-\tau_0}}{\omega + i}. \tag{13}$$

Hence, on the basis of the generalized Liouville theorem (see, for example, ([7])), we obtain that the left- and right-hand sides of (13) are identically equal: a) to zero, if a solution Φ^+ bounded at infinity is admitted; b) to

$$i\sum_{j=0}^{r}\sum_{\nu=0}^{s} C_{j\nu}(i\omega)^j e^{i\omega t_{j\nu}},$$

where $C_{j\nu}$ are arbitrary real constants, $t_{j\nu} > 0$, if solutions Φ^+ are admitted which at infinity have growth of pole type of order not higher than $r+1$. Accordingly, we obtain the following answers:

a)

$$\Phi^+(i\omega) = \int_0^{\infty} k(t)e^{i\omega t}\,dt = e^{-\tau_0};$$

b)

$$\Phi^+(i\omega) = e^{-\tau_0} + \sum_{j=0}^{r}\sum_{\nu=0}^{s}(1-i\omega)\,C_{j\nu}(i\omega)^j e^{i\omega t_{j\nu}}.$$

In the book [2] only solution b) is given. We note that in the class of functions tending to zero at infinity, in the present example the solution of problem (11) does not exist.

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Note: Figure translations are in progress. See original paper for figures.