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Abstract

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THEORY OF QUASI-PRIME NUMBERS

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§ 1. We shall consider, in the interval $(1, x)$, numbers containing only "large" prime divisors p, $p > x^{1/\xi}$, where ξ is a given positive function of x. Such numbers, called below quasi-prime with $\xi = \xi(x)$, were introduced by I. M. Vinogradov (^{1}) in connection with estimates of trigonometric sums over prime numbers, and proved very useful in the consideration of many problems. They were studied by S. Chowla, W. Briggs (^{2}), I. P. Kubilius (^{3}), S. Hooley (^{4}), M. B. Barban (^{5}), and others. Quasi-prime numbers play an especially important role in Yu. V. Linnik' s dispersion method (^{6}) and, in particular, in the solution it gives of the Hardy–Littlewood problem. In this connection, the present work is devoted to the further study of the distribution laws of numbers of this kind. A very general theorem is obtained (Theorem 1), which includes at least all the fundamental questions of the theory of quasi-prime numbers. In particular, from this theorem there follows easily an additive theory of quasi-prime numbers, entirely analogous to the expected theory for the principal binary problems with prime numbers. This theory also extends to the case when one of the (two) quasi-primes is a prime number. The corresponding theorems are natural approximations, both in content and in form, to the hypothetical theorems on prime numbers. The method of their proof uses the Eratosthenes sieve itself, certain modifications of it, the results of combining Yu. V. Linnik' s "large sieve" with his theorems on the density of zeros of Dirichlet L-functions, the results of A. I. Vinogradov (^{7}) on the application of the Riemann ζ-function to numbers with "small" prime divisors, and so on.

Main Theorem 1. *Let an arbitrary sequence of integers be given*

$$a_{\lambda_1},\ a_{\lambda_2}, \dots,\ a_{\lambda_n}, \dots, \tag{A}$$

where λ_n runs through any increasing sequence of positive integers; $S(\lambda_n \leqslant N)$ is the number of terms of (A) with $\lambda_n \leqslant N$, and let some subsequence of prime numbers be given

$$p_1 < p_2 < \dots < p_r, \qquad E_r = \prod_{i=1}^{r} p_i \tag{B}$$

with the condition

$$\prod_{p\leq p_r} p/E_r \ll N^{\sigma}, \qquad \sigma \text{ is a constant} > 0.$$

Then, if for $d \setminus E_r$, $d \leqslant N^{\alpha}$, α is a constant > 0,

$$S_d(\lambda \leqslant N) = \sum_{\substack{\lambda_n \leqslant N \\ a\lambda_n \equiv 0 (\bmod d)}} 1 = \frac{S(\lambda_n \leqslant N)}{f(d)} + R_d(N),$$

$$S_d(\lambda \leqslant N) \ll \frac{S(\lambda_n \leqslant N)}{f(d)},$$

where $f(d)$ is a function, multiplicative on the set of divisors E_r, representable for primes $p \nmid E_r$ in the form

$$f(p) = \frac{p^2}{kp + A} \neq 1$$

with integer $k \geq 1$ and bounded $A = A_p$, then the number $J_r(N)$ of numbers of the sequence (A) with $\lambda_n \leq N$, not divisible by any prime from (B), is expressed by the formula

$$J_r(N) = S(\lambda_n \leq N) \prod_{p \nmid E_r} \left(1 - \frac{1}{f(p)}\right) \left[1 + B \exp\left[-\gamma \frac{\ln N}{\ln p_r} \ln \frac{\ln N}{\ln p_r}\right] + \frac{B}{(\ln N)^{M-25k^2}}\right] + B \ln^M N \sum_{m \leq N^{\alpha}} \mu^2(m) |R$$

where γ is some constant > 0; M, α are arbitrary positive constants; μ is the Möbius function; $B = O(1)$.

The great generality, the degree of nontriviality, and the ease of concrete applications of this theorem, illustrated below by examples, are due to the fact that only uniform distribution "on average" in progressions of difference $d \leq N^{\alpha}$, where α is any small constant > 0, is required of the sequence a_{λ_n}. A very broad class of sequences has this kind of property, including the sequence of prime numbers.

Here we shall confine ourselves only to the statements of the immediate consequences of Theorem 1 and to some comments on them.

§ 2. From Theorem 1, Theorems 2–5 easily follow, which together constitute the additive theory of quasiprime numbers.

Theorem 2. The number $J_{\xi}(N; a, b)$ of representations of an even N in the form

$$N = aq' + bq'' \tag{1}$$

with quasiprimes aq', bq'', $(aq', bq'') = 1$, with $\xi = \xi(N)$, where $\xi(N)$ is any given positive increasing (even arbitrarily slowly) function of N, uniformly in a and b is expressed by the formula

$$J_\xi(N; a, b) = 2\left(\frac{\xi}{e^c}\right)^2 \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \frac{N}{ab\ln^2 N} \prod_{\substack{p>2 \\ p|N}} \frac{p-1}{p-2} \left[1 + R_\xi(N)\right] + BN^\alpha,$$

where c is Euler' s constant;

$$R_\xi(N) = Be^{-\gamma_1 \xi \ln \xi} + Be^{-\gamma_2 \sqrt{\frac{\ln N}{\xi}}} + \frac{B}{\ln^M N};$$

γ_1, γ_2 are positive constants; M, α are any prescribed constants > 0; $B = O(1)$; the products are extended over prime numbers.

The degree of nontriviality of Theorem 2 (and hence also of Theorem 1) is characterized by the following remarks:

A. Among the quasiprimes are all primes $> N^{1/\xi}$, and at the same time there are many fewer quasiprimes on the segment $(1, x)$ than, for example, numbers having only two prime factors, since their numbers $J'_\xi(N)$, $J''(N)$ are expressed respectively in the form:

$$J'_\xi(N) \sim \frac{N}{\ln N} \frac{\xi}{e^c}; \qquad J''(N) \sim \frac{N}{\ln N} \ln\ln N,$$

where one may take $\xi = \xi(N) \to \infty$ arbitrarily more slowly than $\ln\ln N$ (the first estimate is a simple consequence of Theorem 1, the second is known).

B. Theorem 2 gives an asymptotic formula for the number of solutions of additive problems of binary type (1) with a main term of growth down to N^σ, where σ is any small

constant > 0; moreover, in the simplest case $a = b = 1$ the expression $J_\xi(N; a, b)$ differs from that expected in the binary Goldbach problem only by the factor $(\xi/e^c)^2$ instead of 1.

However, one can indicate an equation of the form (1) equivalent to the Goldbach equation in the sense that the principal terms in the growth of the number of their solutions are the same. Namely:

Theorem 3. There exists an integer d, depending on N, such that the number $J(N)$ of solutions of the equation

$$N = q' + q'', \qquad N \text{ even},$$

in quasiprimes $q', q'' = dt + l$ with $\xi = \xi(N)$ as $N \to \infty$ is expressed in the form

$$J(N) \sim 2 \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \frac{N}{\ln^2 N} \prod_{p|N,\ p>2} \frac{p-1}{p-2}. \tag{2}$$

According to Sylvester' s hypothesis, whose validity for almost all even N was established by I. M. Vinogradov' s method (see (^{8})), the number of representations of an even N as the sum of two prime numbers satisfies exactly the same relation (2).

Theorem 4 (quasiprime twins). The number $Q(x, 2)$ of solutions of the equation

$$2 = q' - q''$$

in quasiprimes q', q'' belonging to the interval $(1, x)$ with $\xi = \xi(x) = O(\ln x)$– any positive increasing function of x, is expressed in the form

$$Q(x, 2) \sim 2 \left(\frac{\xi}{e^c}\right)^2 \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \frac{x}{\ln^2 x}.$$

The possibility of replacing here the function ξ, growing arbitrarily slowly with x, by the constant e^c would yield the problem of twin prime numbers.

This theorem extends to the case of any even number instead of 2. However, the following considerably stronger result holds.

Theorem 5. Let $T_\xi(x; D, l; u_1, \dots, u_{m-1})$ denote the number of systems of numbers

$$q,\ q + u_1, \dots,\ q + u_{m-1} \tag{3}$$

with $q \le x$, belonging to the progression $Dt + l$, and such that each number of the system (3) is either prime or a number all of whose prime divisors are greater than x, and let (necessarily)

$$l = l'l''; \qquad l' = \prod_{p_i < x^{1/\xi}} p_i^{\alpha_i}; \qquad l'' = \prod_{p_i > x^{1/\xi}} p_i^{\alpha_i}.$$

Then for any integers $0 < u_1 < \dots < u_{m-1} \le x$, if $m \ge 2$; $u_1 \ge 0$ when $m = 2$; $D \le x^\theta$, where $0 < \theta < 1$, θ is constant; $0 \le l < D$; $(l', D) = 1$, and for any positive function $\xi = \xi(x)$ increasing with x (even arbitrarily slowly), the equality holds

$$T_\xi(x; D, l; u_1, \dots, u_{m-1}) =$$

$$= \left(\frac{\xi}{e^c}\right)^m \frac{x}{D \ln^m x} \prod_p \frac{p - \gamma_m(p)}{p(1-1/p)^m} \left[1 + R_\xi(x)\right] + Bx^\sigma, \tag{4}$$

where c is Euler' s constant; $\gamma_m(p)$ is the number of solutions of the congruence

$$(Dt + l)(Dt + l + u_1) \cdots (Dt + l + u_{m-1}) \equiv 0 \pmod p;$$

p runs through the sequence of prime numbers and

$$R_\xi(x) = Be^{-\gamma_1 \xi \ln \xi} + Be^{-\gamma_2 \sqrt[3]{\ln x/\xi}} + \frac{B}{\ln^m x}; \tag{5}$$

$\sigma, \gamma_1, \gamma_2$ are small constants > 0; $M > 0$ is an arbitrarily large constant.

This assertion is an analogue of the well-known general Hardy-Littlewood hypothesis on prime numbers, the validity of which for almost all systems $u_1, \dots, u_{m-1}$ has also been proved by I. M. Vinogradov' s method (see (9)). We note that the necessity in Theorem 5 of having $R_\xi(x) = Be^{-\gamma\xi \ln \xi}$ for slowly growing ξ arises when attempting to extend Yu. V. Linnik' s dispersion method (6) to certain systems of binary equations with prime numbers.

§ 3. Combining Yu. V. Linnik' s "large sieve" method with his density theorems gives uniformity of distribution "on average" of prime numbers in progressions with difference $d \leqslant N^\alpha$, α being some positive constant. This ensures the nontriviality of Theorem 1 for a sequence of prime numbers. For example, the following holds (see also (10)):

Theorem 6. The number $F(N)$ of representations of an even N in the form

$$N = p + q \tag{6}$$

of the sum of a prime p and a quasiprime q with $\xi = \xi(N)$—any positive increasing (even arbitrarily slowly) function of N—is expressed by the formula

$$F(N) = 2\left(\frac{\xi}{e^c}\right) \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \frac{N}{\ln^2 N} \prod_{p|N,\ p>2} \frac{p-1}{p-2} \left[1 + R_\xi(N)\right] + BN^\sigma,$$

where $R_\xi(N)$ is expressed in the form (5).

We note that equation (6) is the first of Yu. V. Linnik' s equations (see (6), p. 200) for the Goldbach problem.

Along with Theorem 6, one can formulate other results analogous to those given in the preceding paragraph. In particular, one obtains an asymptotic formula for the number of prime numbers $p \leq x$ such that every number of the system

$$p + u_1, \dots, p + u_{m-1} \quad (m \geqslant 1)$$

will be quasiprime with $\xi = \xi(x)$. The corresponding formula differs from formula (4) with $D = 2,\ l = 1$ only by lowering the power $(\xi/e^c)^m$ by one.

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